PRINCIPLES OF

ELECTRIC POWER TRANSMISSION

Wiley Eastern University Edition

FIG. 1. Power transmission in hilly country. 150,000-volt line of the Knoxville Power Co., in North Carolina and Tennessee. The maximum standard span is 2522 feet, and in one section 10 consecutive spans cover four miles of line. *Aluminum Company of America.*

PRINCIPLES OF ELECTRIC POWER TRANSMISSION

BY

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PREFACE TO SECOND EDITION

In the thirteen years since the publication of the first edition, great strides have been made in the art of power transmission, and in power system design, control and operation. Although the fundamental principles of transmission remain the same, improved methods of calculation have been developed which not only result in great savings of time and effort, but enable the engineer to determine with adequate precision many characteristics which before could only be roughly estimated.

The treatment of line reactance calculation has been given a new unity by the use throughout of the method of geometric mean distances. The chapter dealing with steady-state long-line problems has been made more teachable by the inclusion of a preliminary treatment of the direct-current case, and also by the addition of a set of complex hyperbolic function charts. In the derivation of formulas for circle diagrams, the general circuit constants are used in the text, and formulas given whereby these may be expressed in terms of the constants of the equivalent π of the system.

Two new chapters have been added, one dealing with power limits and stability, and the other with the calculation of fault currents. The treatment of mechanical principles has been considerably enlarged.

The recently adopted international meter-kilogram-second system of units has been used in derivations of fundamental formulas for inductance, capacitance, skin effect, potential gradient, etc.

The entire text has been completely rewritten for the second edition, except for a few sections dealing with basic phenomena such as skin effect, in which no advances have taken place.

Grateful acknowledgment is made to Prof. H. B. Dwight, who has collaborated with me in teaching the subject for several years, for suggestions and assistance. Thanks are due to Prof. H. L. Hazen for reviewing parts of the manuscript, and to Mr. Charles Kingsley, Jr., for permission to use his treatment of saturated synchronous reactance.

L. F. W.

February, 1938.

PREFACE TO FIRST EDITION

This work is intended primarily to serve as a text-book for Senior and Graduate students in Electrical Engineering, students who are already familiar with fundamental single-phase and polyphase circuit theory, and with the operating characteristics of the more important types of alternating-current machinery. A familiarity with the use of complex quantities in alternating-current calculations is presupposed.

The book has been in a state of development during several years of teaching the Principles of Electric Power Transmission to Senior and Graduate students of the Massachusetts Institute of Technology, and has been in use as a text in the form of mimeographed notes for some four years. The main educational objective of these courses has been and is the inculcation of the fundamental scientific principles involved in power transmission, and the methods whereby they are made applicable to practical engineering problems; and the secondary objective is the imparting of the maximum practicable amount of information concerning present-day practice in electric power transmission by means of the examples and problems which necessarily form a large part of the work if the students are to gain a thorough mastery of the fundamental principles. Description of apparatus and equipment has been reduced to a minimum consistent with the understanding of the principles involved. A special effort has been made always to impart a clear physical conception of each problem, and wherever a mathematical derivation of considerable length is required, an attempt has usually been made, both before and after the analysis, to present the physical picture independent of mathematical symbols.

For convenience in reference, the most important of the formulas and tables have been collected in an Appendix. Numerous problems are included to afford practice in the application of the principles, and the necessary mathematical tables are appended.

In attempting to correlate the existing knowledge on the subject, naturally many gaps were found. Some of these the author has been able to fill; attention is called to phenomena of which our knowledge is as yet incomplete.

This work was first undertaken in 1922 at the suggestion of Col. T. H. Dillon, now of Harvard University, but then Professor of Electric Power Transmission at the Massachusetts Institute of Technology.

Assisted by the author, he drew up an outline of the course, which has since been modified but slightly. Due to his new work at the Harvard School of Business Administration and to other outside interests, he decided to withdraw from active co-authorship, but his continued interest has been manifested by his constructive criticisms and suggestions. The author wishes to express his sincere thanks for Col. Dillon's aid and cooperation.

Thanks are due also to Prof. Dugald C. Jackson, Head of the Department of Electrical Engineering, Massachusetts Institute of Technology, for suggestions and criticisms, and to Prof. W. H. Timbie, of the Massachusetts Institute of Technology, for valuable suggestions, especially in regard to arrangement of the material. Mr. F. W. Peek of the General Electric Company has very kindly read the chapter on Corona, and has offered valuable suggestions and furnished additional data on this subject. Other engineers have cooperated by allowing the publication of results of their work, and the author desires to express here his appreciation. References to the original sources are given throughout the book.

L. F. Woodruff

Cambridge, Mass., *November*, 1925

PREFACE TO SECOND PRINTING

The principal changes in the Second Printing are the complete rewriting of the chapter dealing with " Mechanical Principles " of sag and stress analysis, so as to include an improved method of calculation which has been developed within the past two years; and second, the correction of the errors which were discovered in the First Edition.

L. F. W.

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PRINCIPLES OF ELECTRIC POWER TRANSMISSION

CHAPTER I

THE GENERAL PROBLEM

1. Importance and Scope of the Electric Power Industry. Economic considerations are the factors which should govern in the development and use of electric power, as in all other engineering problems. Cheap power is the mainspring of the industrial world, on which the safety, comfort and convenience of large populations depend. The development of electric power must proceed in competition with other forms and will replace them in so far as cost, convenience, dependability and safety warrant. The average wage level and standard of living are dependent in very considerable measure on the amount of power used per capita in the industries. Each workman in the United States uses on the average about 5.0 horsepower.

TABLE I

THE ELECTRICAL INDUSTRY IN THE UNITED STATES*

Year	Total Investment		Horsepower Installed in Central Stations
	Central Stations	Electric Railways	
1900	$ 300,000,000	$1,500,000,000	
1905	900,000,000	3,000,000,000	2,800,000
1910	1,800,000,000	4,150,000,000	5,600,000
1915	2,700,000,000	5,000,000,000	10,000,000
1920	4,000,000,000	5,000,000,000	18,000,000
1925	6,300,000,000	5,500,000,000	35,000,000
1930	11,800,000,000	5,500,000,000	43,000,000
1935	13,100,000,000	5,500,000,000	46,000,000
1940			

* Derived principally from the U. S. Census reports and from figures compiled by the Electrical World.

For the transmitting of power to considerable distances, electricity furnishes the only practicable means other than the transport of fuel itself. It is the only practicable means for utilizing water power on a large scale. It saves labor. It is relatively cheap, flexible, convenient, clean and noiseless. It permits the concentration of generating apparatus in large efficient units. It has no competitor except by reason of the relatively large initial investment required.

The enormous growth of the electrical industry in the United States in recent years is indicated by the figures in table I.

2. Prospects for Future Growth and Development. The electrical industry, in almost every phase, presents a most optimistic picture of continued growth, increased usefulness and demand for technical services of high grade. At the beginning of 1910 there were 3 million customers; in 1920, 11 million; in 1930, about 24 million; and in 1938 more than 26 million. The power consumption per customer also has increased at a rapid rate. Not only are the older electrical devices continually obtaining broader adoption, but there are continually being produced new types of electrical devices for various purposes, all of which go to swell the demand for electrical energy. It is very rare to find an electrical device for performing any function replaced by a non-electrical device. The efficiency of generating stations is increasing. In steam plants, the average number of pounds of coal burned per kilowatt-hour generated in 1920 was 3.0; in 1925 it was 2.1; and in 1930 it was 1.6. The most efficient stations are now operating with only about half this fuel consumption, and so it is certain that the average downward trend will continue for some time to come even without any new developments, as the older plants are replaced by more modern equipment. Improvements in efficiency in hydroelectric equipment have also been considerable in recent years, but it has not been possible to have them comparable with improvements in steam plants for the reason that the efficiencies of hydraulic turbines have been around 95 per cent for several years, and the additional savings possible are comparatively small. For several years past, steam plants have been generating about 60 per cent of the total electrical energy, to about 40 per cent for the water-power stations. The ratio of installed capacity, steam to water power, is about 70 per cent to 30 per cent. The water-power plants are operated at a higher load factor than the steam plants, because at times of light load it is more economical to shut down the steam plants to save coal, than to shut down the hydroelectric plants to save water. Labor costs also are less in the hydroelectric plants.

At present about 95 per cent of the urban homes in the United States are wired for electricity, 60 per cent of those in small towns, and 26 per cent of farms having dwellings valued above $500. The use of small electric labor-saving devices and other appliances should be several times greater than at present.

The increased activity of each industry in the country, and in the world, has its effect on the electrical industry, which is most intimately bound up with the prosperity of the entire country.

Taken in connection with electrical manufacturing and the independent installations, it appears that the electrical industry has been more than doubling every ten years. There is every prospect for a rapid growth for a long time to come.

The U. S. Geological Survey estimates that there are available in the United States about 50,000,000 kilowatts of water power, of which 12,000,000 had been developed by 1938. The development of a large portion of the remainder within the next two or three decades will, without much doubt, be accomplished as the result of the rising prices of coal and labor, or as a by-product of the projects for flood control, irrigation and the prevention of soil erosion.

3. Present Tendencies toward Interconnection and Consolidation. The present tendency continues to be toward the abandonment of the small steam station, the more complete development of hydroelectric power and the interconnection of the large steam and hydro stations by high-voltage networks. The development of long-distance transmission has been largely a question of practicable voltages. More recently, much attention has been given to the possibilities of direct-current transmission.

The same period of years has seen the growth of size of generating units from a few kilowatts to more than 100,000 kva and of central stations to over a million kilowatts. The efficiencies have increased so that, instead of using 4 or 5 pounds of coal per kilowatt-hour, the large modern central stations are using as little as 0.9 pound per kilowatt-hour.

Table II shows the growth in the size of commercial transmission voltages.

Like many other businesses, the electric power industry shows more economical operation in large units than in small ones. This is true for a variety of reasons. A station supplying only one customer must have installed a generating capacity equal to the maximum demand of the customer plus a suitable reserve to be used in case of accident, repairs and emergency. When the number of customers is large, the probability that their maximum demands will occur at the same time is small. In general, the greater the number of customers, the smaller is their *diversity factor*, and the larger may be the ratio between the station capacity and the total connected load. (*Diversity factor* is defined as the quotient of the total of the maximum powers, irrespective of time of occurrence, of all the loads, divided by the maximum of the total power of all the loads.) The *load factor* of large stations is usually greater than that of small ones. (The *load factor* for any period is defined as the quotient of the average load for the period divided by the maximum load of the period. In practice, the maximum load used in computing load factor is ordinarily taken as the greatest average load for a definitely specified period, such as 5 minutes or 30 minutes.) In general, the diversity factor is greater than unity; the load factor is less than unity. The efficiency of large units is greater, and their cost per unit of capacity, taking into consideration necessary accessory equipment and housing, is smaller. In large stations the economies which

TABLE II

Year	Highest Commercial Transmission Voltages
1889	4 kilovolts
1892	10
1893	11
1896	25
1900	60
1908	110
1913	150
1923	220
1937	287

may be effected by the use of labor-saving devices such as automatic stokers and coal- and ash-handling apparatus are very large, and there is an important saving in both labor and overhead over the same total capacity of stations in smaller units. Fuel and other supplies may be purchased more economically in large quantities. Large companies may obtain capital more readily to finance extension and provide power service impossible to small companies. Expert technical, financial and legal advice may be obtained at a decreased cost per unit of capacity. In very large power companies, the capacity of the reserve generating equipment required may be a smaller percentage of the total installed capacity than it is for small companies. In general the reserve capacity must be at least sufficient so that the maximum load can be supplied with one of the largest units out of commission for repairs.

The desirability of a more extensive use of hydroelectric power needs no argument. The question is one of markets and costs. The initial expense is relatively high and is justified under present conditions only where the project is particularly favorable. As the power demands grow and fuel becomes more expensive, the less favorably situated water power will ultimately be developed.

The interconnection of hydro and steam power stations offers opportunity to effect economies in operation. Water supply is generally fluctuating, and the steam stations may be utilized as reserves during unfavorable water conditions. Also, it often happens that periods of greatest drought in one watershed do not coincide with those of a neighboring watershed, and mutual benefits may be derived from an interchange of power at favorable times. This fact constitutes a potent argument against the wisdom of the enactment of stringent laws designed to prohibit the export of power from one state to another, or from one country to another.

The utilization of large water powers usually requires long transmission lines which are expensive in first cost. Under present conditions and practicable voltages these long lines are economical only when transmitting large blocks of power at high daily and annual load factors.

The tendency is toward universal electrification through combinations of water and steam power and comprehensive transmission networks. The combination insures stability of supply and price and will tend to make price more nearly independent of locality and permit a more widespread location of our industries and industrial communities. The advantages of such a system were very well expressed by the late General Guy E. Tripp, formerly Chairman of the board of the Westinghouse Electric and Manufacturing Company, in an article in the World's Work, as follows:

"It will enable us to make the fullest use of labor-saving machinery in every industry, thereby increasing the incomes of the workers and decreasing the cost of the products of labor for the consumers.

"It will permit the electrification of many of our railroads and will also reduce to a minimum the tonnage of fuel to be hauled. For both of these reasons, the efficiency of our transportation system will be greatly improved.

"It will, by making industry and comfortable living conditions possible almost anywhere, tend to draw the people out of the congested cities and distribute them throughout the country.

"It will provide the farmer with an ideal means of doing a large part of his work and give him welcome relief from his labor difficulties. It will also serve him by greatly reducing the price of fertilizer; for with ample supplies of cheap power, fertilizer can be manufactured at an exceedingly low cost.

"Not the least of its benefits will be the lightening of the tasks of the housekeeper by making possible the universal use of electric cooking and labor-saving devices.

"It will also aid our national defense. One of the greatest difficulties experienced during the late war was to secure sufficient power to produce essential materials in the vast quantities required. Almost all of our power plants were small and scattered, and there was no way of combining their outputs so as to make large-scale production possible. With a single superpower system in operation, however, any proportion of its capacity can be allotted to war work in emergencies. Ample power thus being obtainable, war industries of any size can be brought into being with maximum speed and operated with a minimum amount of labor. They can, moreover, be located wherever desired, thus making it possible to avoid a repetition of that cardinal military error — the production of the major portion of our war supplies in the most vulnerable parts of the country."

4. Requirements of Satisfactory Service. A central station should deliver through its transmission and distribution systems to the loads electric power characterized by constant or nearly constant voltage, dependability of service, balanced voltage (if polyphase), efficiency such as to give minimum annual cost, constant frequency and sinusoidal wave form (if alternating current), and freedom from inductive interference with telephone lines.

5. Voltage Regulation. The amount of variation in voltage which is allowable depends to a considerable extent on the kind of service which is being supplied. A voltage variation has a large effect upon the operation of both power machinery and lights, but in sparsely settled districts it is not practicable to install regulating devices sufficient to make the voltage variation as small as it could well be made in a densely populated city.

The effect of voltage variations on Mazda lamp characteristics is shown in table III. A given percentage change in voltage causes a very much greater percentage change both in candlepower and in life. In general, good practice does not allow more than a 5 per cent variation of voltage in lighting circuits.

A motor is presumably designed to have its best characteristics at the rated voltage, and consequently a voltage that is too high or too low will result in a decrease in efficiency. In both synchronous and induction motors, the breakdown torque varies rapidly with the voltage. Too low a voltage therefore brings with it the danger of the machines falling out of step, and it also causes them to draw more current from the line. In general the copper losses are increased more than the amount of the decrease in core loss due to the lower flux density attendant upon the lower voltage. On the other hand, if the voltage is too high, the core loss increases more than the copper loss is decreased in most cases. In a synchronous motor, a voltage greater than normal means either that a larger field current has to be used or that the power factor will be more lagging. This effect may be considerable if the motor is operated at a fairly high point on its magnetization curve.

If the fluctuations in the voltage are sudden, in the current wave there will be attendant fluctuations of much greater amplitude, and these may cause the tripping of circuit breakers and consequent interruptions to service. These fluctuations also have a tendency to start hunting in synchronous machinery.

TABLE III

PROPERTIES OF MAZDA LAMPS AS FUNCTIONS OF VOLTAGE

Per Cent Rated Voltage	Per Cent Normal Candlepower	Per Cent Normal Life
90	70	400
95	83	170
100	100	100
105	119	50
110	139	30
115	160	18

Usually the voltage at the generator terminals is held constant by automatic regulators. Where this is done, in some cases the voltage variations at the load may be made sufficiently small by keeping the resistance and reactance of the lines and feeders low. Generally, however, it is impracticable to keep the regulation sufficiently good by this method, and special apparatus must be installed to regulate the voltage.

6. Dependability. The losses which purchasers of power sustain when shutdowns are caused by failures of their electric power supply are usually vastly greater than the actual value of the power they would use during this period, on account of the expense of idle workmen and machines and other overhead charges, and in some cases ruined material. A shutdown of only one minute would probably cause at least a ten-minute waste of time, owing to the natural delay in getting under way again. Sometimes it is very difficult to get a system in operation again after a complete shutdown, on account of the large starting currents taken when numbers of the customers try to start up their motors at once. Precautions have to be taken to prevent the blowing of the circuit breakers during the period of restarting. Usually starting can be accomplished by waiting a short time between the connecting in circuit of the different feeders of the primary distribution system. If the secondary distribution system is tied together, however, additional difficulties arise.

Interruptions to service cause irritation and loss of good will, and are sometimes positively dangerous to life and property. Failure of power in hospitals, and in crowded theaters and stores, may lead to very grave consequences. In addition to the monetary value of what is technically termed good will, there is a moral and sometimes a franchise or contractual obligation to keep the systems going and to furnish uninterrupted service.

Dependability of service depends upon the use of strong mechanical construction of towers, lines, etc., with ample provision for the effects of wind, sleet, ice, vibration and the pull of broken wires; the use of parallel lines and feeders and the frequent testing of insulators and cables; in the power house, the installation of current-limiting reactors, strong bus construction, duplicate switching apparatus and double or ring construction of the bus. There should be ample relay protection for isolating faults, ground wires and lightning arresters to provide paths to ground for abnormally high charges on the lines, a reasonable amount of spare apparatus and trouble crews always ready to do emergency repair work in minimum time. Interconnections with other systems are a great aid in times of trouble. Most important of all is the use of high-grade machinery and insulation, and care to avoid overheating with its attendant rapid deterioration of insulation. The stability characteristics of the system should be satisfactory.

If the principal source of power in a system is hydraulic, periods of low flow must be provided for by standby steam or diesel plants, or by interconnection with other systems.

7. Unbalance. If an unbalanced polyphase voltage is supplied to a customer operating synchronous or induction motors, it will result in a decrease in the efficiency of his machinery, and also a decrease in its maximum power output. If the motors are called upon to deliver full load when their terminal voltages are unbalanced, therefore, they are liable to considerable damage from overheating. The balance of the voltage may be maintained by having only balanced loads connected to the circuit and by using symmetrical transformer connections and symmetrical spacing of the line conductors, or else proper transposition. When unbalanced and single-phase loads have to be taken care of, in some cases it may be advisable to run separate feeders for this service. If not, customers having heavy unsymmetrical loads may be penalized by being charged a higher rate. The consequent unbalancing effect on the voltage triangle can be reduced to a minimum by means of special phase-balancing apparatus and by the exercise of care and thought in the connecting of various unbalanced and single-phase loads to such phases as to produce a minimum total unbalance of current.

8. Efficiency. The efficiency of a transmission system is not of much importance in itself, the important economic feature of design being the layout of the system as a whole so as to perform the requisite functions of generating and delivering power with a minimum overall annual cost, due consideration of course being paid to future expansion and possible eventual economies which may be effected by providing to some extent beforehand for such expansion. The power losses in lines and machinery are but one item, albeit an important one, of the total annual cost. The losses are affected to a very considerable extent by the power factor, so it is only fair that customers having loads of low power factor should be penalized by being charged at a higher rate per kilowatt-hour than are those who take power at high power factors. Loads of low power factor also require greater generator capacity than those of high power factor, for the same amount of power, and produce larger voltage drops in the lines and transformers.

9. Other Miscellaneous Requirements. Frequency must be maintained constant so that manufacturing operations which are carefully standardized will not be interfered with by the change in motor speed which would accompany a change in frequency. In nearly all large systems the frequency is kept so constant that synchronous clocks may be operated directly from the power mains.

The alternating voltage supplied to customers should have a sine wave form, as any harmonics which might be present would have a deleterious effect upon the efficiency and maximum power output of the connected machinery. Harmonics may be avoided by using generators of good design and by the avoidance of corona and of high flux densities in the transformers. Some harmonics may be introduced by the loads themselves, as in arc furnaces and arc welding apparatus. The harmonic currents drawn by such loads cause line drops at these frequencies, and so there may be harmonics in the load voltage even though there be none in the generated voltage.

Power companies have the responsibility of keeping to a practical minimum the electrostatic and electromagnetic field disturbances which they set up at a distance from their lines, on account of the fact that these fields tend to cause objectionable noises and hums in apparatus connected to communication circuits. Interference with telephone lines running parallel to power lines may be avoided by limiting as much as possible the amount of zero-sequence and harmonic current flowing, particularly multiples of the third harmonic, and by the proper transposition of both power lines and telephone lines.

The choice of the best voltage and frequency for a given project is a very important matter, as is the decision regarding the type of distribution wiring layout to be used. Three-phase is nearly standard in power work, and gives advantages in transmission and machine design. The frequency of 60 cycles per second is becoming more and more the standard frequency of the country, although there are some installations of 50 cps, and a considerable number of power circuits and primary distribution circuits in cities where direct-current lighting service is supplied operate at 25 cps. New York City has such a system. There are one or two systems which operate at 40 cps. However, electric lights do not operate satisfactorily at frequencies lower than 50 cps, as the lower frequencies cause a noticeable and disagreeable flicker in the light. The higher frequencies also have the advantage in that they offer a much greater range of speed variation for synchronous and induction motors. For instance, 60-cps synchronous or induction motors may be designed for nominal speeds of 3600, 1800, 1200, 900, 720, 600, 514, 450, etc., revolutions per minute, while the 25-cps motors of these types are confined to 1500, 750, 500, 375, 300, etc., rpm. Twenty-five-cycle-per-second transformers cost from 40 per cent to 65 per cent more than 60-cps transformers of the same rating, owing to the smaller number of flux linkages required at the higher frequency. There is also a slight difference in efficiency in favor of the 60-cps transformers. On the other hand, the reactance of lines and machinery at 60 cps is 2.4 times as great as at 25 cps. Machinery for operation at 60 cps usually requires fewer turns than 25-cps machinery, which makes its reactance relatively somewhat less.

10. Essential Features of Transmission Systems. In the United States there are well over 1000 commercial power transmission lines operating at voltages of 44,000 and higher. All these are three-phase alternating-current, and the great majority operate at a frequency of 60 cps.

The principal elements of a high-voltage power transmission line are the following:

1. The conductors, usually three in number for a single-circuit line and six for a double-circuit line. The usual materials are copper or aluminum, which may be reenforced with steel. The conductors may be solid or stranded. On very high-voltage lines hollow conductors made of either sectors or strands are not uncommon.

2. Step-up and step-down transformers at the sending and receiving ends, respectively. The purpose of the transformers is to provide

for a sufficiently low current on the line itself so that ohmic losses and line voltage drop will not be excessive.

3. Line insulators, which must mechanically support the line conductors and isolate them electrically from ground potential. Insulated cables may be used but are much more expensive than overhead lines for cross-country work.

4. Supports, which may be steel towers, wood poles or frames or concrete poles.

5. Protective devices, including:

(*a*) Ground wires suspended above the power conductors and grounded at each tower or pole. The function of the ground wire is principally to intercept direct lightning strokes and conduct the charge to the nearest ground connections. It also affords some protection against voltages induced by lightning near by; it adds mechanical strength, and by providing a return path for zero-sequence current reduces inductive interference with parallel communication lines. The presence or absence of the ground wire will affect the magnitude of short-circuit currents.

(*b*) Lightning arresters, which have the characteristic of safety valves. They act as insulators at voltages up to and slightly exceeding normal, but at higher voltages they become good conductors and quickly drain the excess voltage from the line.

(*c*) Circuit breakers, which are used both for normal control of the circuits and for automatic isolation of faulty sections of the system in times of trouble.

(*d*) Relays of various types for the automatic control of the circuit breakers. Some types require special pilot wires or a carrier-current channel; others operate directly under the control of the voltage, current, power and angle characteristics of the power circuit.

(*e*) Counterpoises or artificial grounds used particularly in locations where the soil is very poorly conducting.

6. Voltage-control devices, including:

(*a*) Synchronous reactors at the receiving end of the line.
(*b*) Tirrill or other regulators for the generators and reactors.
(*c*) Induction regulators.
(*d*) Tap-changing transformers.
(*e*) Line-drop compensators.

7. Miscellaneous equipment for the control of the system as a whole, such as quick-response excitation for improving stability; governors and

frequency control apparatus for controlling also load division among different generating stations; communication equipment, power-limiting reactors.

11. Economic Justification for Construction of a Line. The construction of a power transmission line may be justified economically by the existence at a reasonable distance apart of a load center and a more favorable generating site; of two systems or portions of systems each requiring standby equipment for assuring continuity of service; of neighboring systems which would benefit by an interchange of energy at different seasons; or by any combination of these conditions.

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CHAPTER II

INDUCTANCE AND REACTANCE

The inherent voltage regulation of a transmission or distribution system depends on the characteristics of the generator and transformer machinery, and also largely on the resistance and inductance of the lines themselves. The inductance of a transmission line depends on the material, the dimensions and configuration of the wires and the spacing between them, and the calculation of the inductance is based on the following fundamental definitions.

12. The MKS System of Units. A system of units based on the meter, the kilogram and the second was approved by the International Committee of Weights and Measures in 1935 for adoption in 1940. This system will be used in the derivations of inductance and capacitance formulas in this book.

The **ampere** is the constant current which, maintained in two straight thin conductors of infinite length separated by a distance of 1 meter, produces between the conductors a force of 2×10^{-7} mks unit of force (joules per meter) per meter of length. The symbol for current is I or i.

The **volt** is the difference in electrical potential between two points of a conductor carrying a constant current of 1 ampere when the power dissipated between these points is equal to 1 watt. The symbol for potential difference is E or e.

The **coulomb** is the quantity of electricity transported each second by a current of 1 ampere. The symbol for charge or quantity is Q.

The **ohm** is the electrical resistance between two points of a conductor when a constant difference of potential of 1 volt, applied between these points, produces a current of 1 ampere. The symbol for resistance is R or r.

The **weber** is the magnetic flux which, linking a circuit of a single turn, would produce an electromotive force of 1 volt if brought to zero in 1 second at a uniform rate. The symbol for flux is ϕ.

The **henry** is the inductance of a closed circuit in which an electromotive force of 1 volt is induced when the current traversing the circuit varies uniformly at the rate of 1 ampere per second. The symbol for self-inductance is L.

The **farad** is the electrical capacitance between two conducting bodies between which appears a potential difference of 1 volt when they are charged with 1 coulomb of electricity each, of opposite signs. The symbol for capacitance is C.

Magnetic flux density, in webers per square meter, is given the symbol B.

One additional fundamental unit is required besides the meter, kilogram and second; at the time of writing this has not been selected.

Unit magnetomotive force is 1 ampere-turn. The symbol is the script $\mathscr{F}$.

Unit magnetizing force or **unit magnetic field intensity** is 1 ampere-turn per meter. The symbol is H.

Space permeability is numerically the ratio between flux density and field intensity in free space, and is $4\pi 10^{-7}$ or 1.257×10^{-6}. The symbol for space permeability is μ_0.

Relative permeability μ/μ_0 is the ratio between the actual permeability of a substance and that of free space.

The number of **flux linkages** in a circuit will be defined as the summation or integral of all the elements of flux multiplied by the *fraction* of the total current linked by each. The symbol for flux linkages is λ.

Inductance is the rate of change of flux linkages with current, and the voltage of self-induction is $L \frac{di}{dt}$ volts, which is the same as $\frac{d\lambda}{dt}$. A somewhat looser definition of inductance, which is exact only when the permeability of the magnetic circuit is constant, is the number of flux linkages divided by the current. Using this second definition, we would say that the inductance of a circuit in henries is equal to the number of flux linkages in the circuit divided by the current of the circuit, there being no extraneous flux set up by other means than the circuit itself.

13. Magnetic Field about a Long Straight Round Wire. If a long, straight, uniform, isolated, round wire is carrying current, it is evident from considerations of symmetry, and it may also be shown experimentally with iron filings or a compass needle, that the lines of flux set up by the current will be arranged in concentric circles around the wire. From this it follows that at any fixed radius from the center of the wire the flux density and the magnetizing force are uniform. This fact enables us to write almost immediately the expression for the flux density for points outside the wire, in the following manner. Referring to figure 2, which represents a wire of radius r carrying a steady current I amperes, the magnetomotive force around the wire is I ampere-turns.

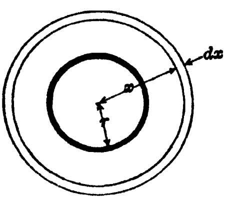


FIG. 2. Elementary flux path outside an isolated round wire.

Along the elementary flux path of radius x meters, the magnetic field intensity is equal to the total magnetomotive force $\mathcal{F}$ divided by the length of path, which is $2\pi x$.

$$H = \frac{I}{2\pi x} \text{ amp-turns per m.} \tag{1}$$

The magnetic flux density along this path, in free space, is

$$B = H\mu_0 = \frac{2 \times 10^{-7} I}{x} \text{ webers per sq m.} \tag{2}$$

Considering now a line of flux inside the wire, as illustrated in figure 3, the symmetry of the figure again leads us to the conclusion that the lines of flux are arranged in circles concentric with the wire itself. We can write, therefore, by analogy to (1),

$$H = \frac{I_x}{2\pi x} \text{ amp-turns per m,} \tag{3}$$

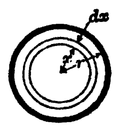


FIG. 3. Elementary flux path inside an isolated round wire.

where I_x is the current in the portion of the wire enclosed by the circle of radius x. If the current in the wire is uniformly distributed over the cross section, then I_x is manifestly equal to the total current I multiplied by x^2/r^2. Substituting for I_x in (3), we have

$$H = \frac{Ix}{2\pi r^2} \text{ amp-turns per m;} \tag{4}$$

and for the flux density,

$$B = H\mu = \frac{2 \times 10^{-7} Ix}{r^2} \frac{\mu}{\mu_0} \text{ webers per sq m.} \tag{5}$$

Figure 4 shows a plot of flux density against distance from the center of a round wire of non-magnetic material. The straight line representing the flux density inside the wire and the hyperbola representing the density on the outside form a continuous curve, although the slope is of course discontinuous at the surface of the wire.

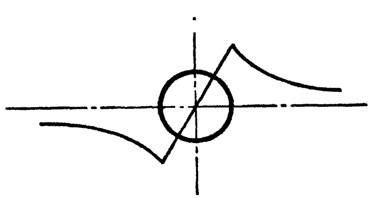


FIG. 4. Plot of flux density (ordinates) against distance from axis (abscissas) for an isolated round wire of non-magnetic material.

Figure 5 shows another way of indicating the flux distribution about a round wire. The amount of flux per unit length contained between any two adjacent concentric cylinders (indicated by circles in figure 5) is the same. This means that, if we should lay off abscissas in figure 4 equal to the radii of the circles of figure 5, and erect ordinates through these points, these ordinates would divide the flux density curve into equal areas.

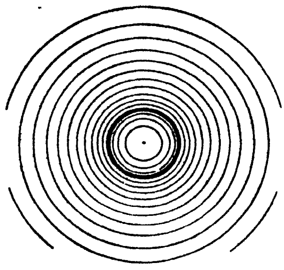


FIG. 5. Magnetic field about an isolated round straight wire carrying current.

If the relative permeability of the wire had been greater than unity the two curves of figure 4 would not join. In iron wires the permeability is not a constant at different radii in the wire owing to the different degrees of saturation that obtain, and so the curve of flux density inside the wire is not a straight line. In treating this case analytically it is usual to assume an average value of permeability for the whole wire. This method, though not exact, is usually sufficiently accurate for practical purposes.

14. The Inductance of a Return Circuit of Two Parallel Wires. In a medium of constant permeability, B and H are proportional, and from this fact it follows that we can find the total flux set up by two or more independent magnetizing forces by adding (vectorially) the individual fluxes set up by each of the fields acting separately, as well as by the more evident way of first combining the two forces at each point and obtaining the resultant force there, and then from the resultant force calculating the total flux.

We will assume that the two long equal parallel wires whose cross sections are shown in figure 6 form a return circuit for a current of I amperes and that they are surrounded by a medium of relative permeability unity. We will consider separately the component of flux set up by the current in each wire as justified above. If the wires are of magnetic material, a small error is introduced because of the fact that the flux set up by one wire is slightly warped from its circular form owing to the presence of the magnetic material of the other conductor. If the separation of the wires is several times their radius the error introduced is exceedingly small.

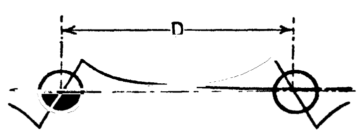


FIG. 6. Plot of magnetic flux density against distance. Return circuit of two parallel round wires of unit relative permeability and radius r meters.

The flux set up by the current in the wire on the left in figure 6 may be divided into four parts: (1) that inside the wire itself; (2) that passing between the wires; (3) that having a path crossing the other wire; and (4) that surrounding both wires. Correspondingly the flux set up by the current in the other wire may be divided into four parts. We wish to find the inductance of the circuit per meter of length. This constant in henries is equal to the number of flux linkages per meter divided by the current I. Let the origin be taken at the center of the wire on the left. Considering first only the flux caused by the wire on the left inside itself, we find from equation 5 that the flux per meter length in a ring of radial thickness dx and radius x is equal to $\frac{\mu}{\mu_0}\frac{2 \times 10^{-7} Ix\,dx}{r^2}$ webers per meter. This flux surrounds the fraction $\frac{\pi x^2}{\pi r^2}$ of the total current. The number of flux linkages due to this internal flux is therefore equal to

$$\frac{\mu}{\mu_0}\int_0^r \frac{2 \times 10^{-7} Ix^3\,dx}{r^4} = \frac{\mu}{\mu_0}\frac{10^{-7} I}{2} \text{ linkages per m.} \tag{6}$$

The flux surrounding both wires does not encircle any net current and hence does not contribute any flux linkages to the total. The flux which has part of its path through the other wire surrounds on the average about half the current, some of this flux surrounding nearly all the current, and some surrounding hardly any. The flux passing between the two wires surrounds the total current of one wire, and in integrating this flux we will integrate out to the center of the other wire, thus taking into account, approximately at least, the linkages caused by the flux having part of its path through the right-hand wire. It will be shown later that this gives an exact result for non-magnetic wires. Since the flux density outside the wire and distant x meters from its axis is $\frac{2 \times 10^{-7} I}{x}$ webers per square meter, the number of linkages due to the flux outside the wire is

$$\int_r^D \frac{2 \times 10^{-7} I\,dx}{x} = 2 \times 10^{-7} I \ln\frac{D}{r} \text{ linkages per m.} \tag{7}$$

The total number of linkages per meter length about one wire is equal to $10^{-7} I\left(2 \ln\frac{D}{r} + \frac{1}{2}\frac{\mu}{\mu_0}\right)$ and the inductance per meter length of one wire is this divided by the current, or

$$L = 10^{-7}\left(2 \ln\frac{D}{r} + \frac{1}{2}\frac{\mu}{\mu_0}\right) \text{ henries per m, for one wire.} \tag{8}$$

Since a mile contains 1609.4 meters, we may write

$$L = 0.7411 \log_{10}\frac{D}{r} + 0.08047\frac{\mu}{\mu_0} \text{ millihenries per mile, for one wire.} \tag{9}$$

The inductance is seen to be the sum of two terms, the first of which depends on the size and spacing of the wires, and the second of which depends only on the permeability. In ordinary overhead lines, the second part is much smaller than the first, unless the conductors are made of iron or steel. Decreasing the spacing or increasing the size of conductors will have the effect of decreasing the amount of inductance per unit length, and vice versa.

15. General Expression for Linkages in a Group of n Parallel Wires. The expression for flux density outside an infinitely long wire, as given in (2), may be integrated from r to ∞ to yield an expression for linkages due to external flux, for each unit length of the wire. The resulting expression, however, is infinite, and so it is desirable, in computing linkages about one of an arbitrary group of parallel wires, to select a method of attack which will avoid indeterminate expressions involving the differences of infinite terms.

In figure 7 is represented a group of n parallel round wires carrying current, and it is stipulated further that these wires comprise the complete circuit at the section shown, or in other words the sum of the currents in the

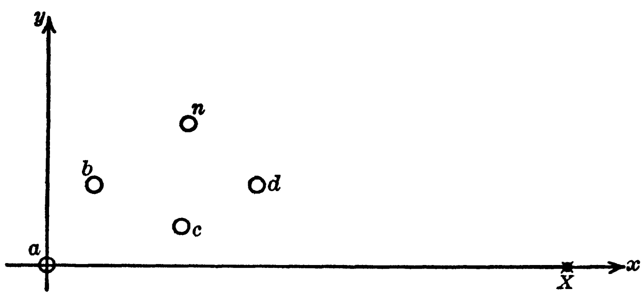


FIG. 7. Arbitrary group of n parallel wires carrying current.

n wires is equal to zero. Axes are set up as shown, through the center of a wire a about which the linkages are to be computed. Mark any point X on the x axis more distant than a from any and all of the other wires.

The letter I with appropriate subscripts will be used to designate the various currents, and D with appropriate subscripts is the symbol for the various distances involved.

The number of flux linkages about wire a due to its own current I_a, and produced by lines of flux which cross the x axis between the origin and X, is equal to

$$10^{-7}\left(\frac{I_a}{2}\frac{\mu}{\mu_0}+\int_{r_a}^{D_{aX}}\frac{2\,I_a\,dx}{x}\right)=10^{-7}\left(\frac{I_a}{2}\frac{\mu}{\mu_0}+2\,I_a\ln\frac{D_{aX}}{r_a}\right)$$

linkages per m. (10)

The number of lines of flux about wire a due to the current I_j in any other one j of the remaining wires $b, c, d, \cdots n$, and produced by lines of flux which cross the x axis between the origin and X, is equal to

$$10^{-7} \int_{D_{ja}}^{D_{jX}} \frac{2\, I_j\, dx}{x} = 2 \times 10^{-7}\, I_j \ln \frac{D_{jX}}{D_{ja}} \text{ linkages per m.} \qquad (11)$$

The total number of linkages produced by flux which crosses the x axis between the origin and X is equal to

$$10^{-7} \left(\frac{I_a}{2} \frac{\mu}{\mu_0} + 2\, I_a \ln \frac{D_{aX}}{r_a} + 2\, I_b \ln \frac{D_{bX}}{D_{ab}} + 2\, I_c \ln \frac{D_{cX}}{D_{ac}} + 2\, I_d \ln \frac{D_{dX}}{D_{ad}} + \cdots + 2\, I_n \ln \frac{D_{nX}}{D_{an}} \right) \text{ linkages per m.} \qquad (12)$$

Since it has been postulated that the sum of all the currents is zero, we may write

$$I_n = -I_a - I_b - I_c - \cdots - I_{n-1}. \qquad (13)$$

This value of I_n is now substituted in the last term of (12), thus forming a number of new terms which may be combined with the other terms in (12) involving logarithms. The result for linkages due to flux out to X is

$$10^{-7} \left(\frac{I_a}{2} \frac{\mu}{\mu_0} + 2\, I_a \ln \frac{D_{an}}{r_a} \frac{D_{aX}}{D_{nX}} + 2\, I_b \ln \frac{D_{an}}{D_{ab}} \frac{D_{bX}}{D_{nX}} + \cdots + 2\, I_{n-1} \ln \frac{D_{an}}{D_{a(n-1)}} \frac{D_{(n-1)X}}{D_{nX}} \right) \text{ linkages per m.} \qquad (14)$$

Now let X approach ∞, and it may easily be shown that all the fractions involving X in (14) approach unity as a limit. The actual total number of linkages λ_a about wire a, then, is equal to

$$\lambda_a = 10^{-7} \left(\frac{I_a}{2} \frac{\mu}{\mu_0} + 2\, I_a \ln \frac{D_{an}}{r_a} + 2\, I_b \ln \frac{D_{an}}{D_{ab}} + \cdots + 2\, I_{n-1} \ln \frac{D_{an}}{D_{a(n-1)}} \right) \text{ linkages per m.} \qquad (15)$$

The common numerators D_{an} in the logarithmic terms of (15) may be separated out, and by a reverse application of relation (13) they unite into a single new term $2\, I_n \ln \frac{1}{D_{an}}$. We have then finally

$$\lambda_a = 10^{-7} \left(\frac{I_a}{2} \frac{\mu}{\mu_0} + 2\, I_a \ln \frac{1}{r_a} + 2\, I_b \ln \frac{1}{D_{ab}} + 2\, I_c \ln \frac{1}{D_{ac}} + \cdots + 2\, I_n \ln \frac{1}{D_{an}} \right) \text{ linkages per m.} \qquad (16)$$

This may be written in the form

$$\lambda_a = 0.000\,080\,47\, I_a \frac{\mu}{\mu_0} + 0.000\,741\,1 \left(I_a \log_{10} \frac{1}{r_a} + I_b \log_{10} \frac{1}{D_{ab}} + I_c \log_{10} \frac{1}{D_{ac}} + \cdots + I_n \log_{10} \frac{1}{D_{an}} \right) \text{ linkages per mile.} \qquad (17)$$

16. Geometric Mean Distance. Formulas (16) and (17) are useful in themselves, but they may also be considered as marking the completion of one important step in the development of a very useful method of computing inductance, by *geometric mean distances.*

Figure 8 represents the cross section of two parallel cylindrical nonmagnetic conductors of arbitrary cross section. Let the cross section of each conductor be divided into a large number of parts n, such that equal currents flow in all the individual parts. If the current density is uniform in the conductors, then the parts will be of equal size in each conductor.

If the current density is not uniform, the cause of the non-uniformity must be either skin effect or heterogeneity of the material. These cases will be discussed later.

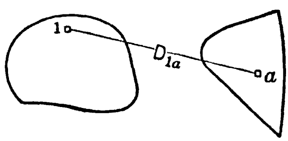


FIG. 8. Parallel cylindrical conductors of irregular section.

The small individual areas on the left-hand conductor will be designated by 1, 2, 3, 4, etc., to N; and those on the right by a, b, c, d, etc., to n. Only one of each is shown in the figure. It is postulated that the left-hand conductor carries a total current I, and the right-hand conductor a current $-I$. Then the current in each element will have a magnitude I/n, and will be positive for elements in the left-hand conductor and negative for those in the right-hand conductor.

We may apply (16) to the present case just as if the various elements were cross sections of independent conductors. The number of linkages about element 1 will be

$$\lambda_1 = 10^{-7} \frac{2I}{n} \left(\frac{1}{4} + \ln \frac{1}{r_1} + \ln \frac{1}{D_{12}} + \ln \frac{1}{D_{13}} + \cdots + \ln \frac{1}{D_{1N}} - \ln \frac{1}{D_{1a}} - \ln \frac{1}{D_{1b}} - \ln \frac{1}{D_{1c}} - \cdots - \ln \frac{1}{D_{1n}} \right) \text{ linkages per m.} \qquad (18)$$

If filament 1 is not round, then the value of r_1 must be taken as an equivalent filament radius, which may be selected such that the constant term is left unchanged.

Similar expressions may be written for linkages about filaments 2, 3, 4, etc., of the left-hand wire, and the sum of the linkages about all the filaments from 1 to N will be equal to the number of linkages about the entire left-hand conductor. However, we must carefully note that a linkage has been defined in a way that makes it dependent upon the circuit to which it is referred, and one linkage about one of n equal filaments of a conductor would contribute only $1/n$ of a linkage with respect to the entire conductor. Therefore, we must multiply the sum of the individual filament linkages by a factor $1/n$ to obtain the total conductor linkages with respect to the entire circuit. The number of linkages λ about the left-hand conductor is

$$\begin{aligned}\lambda = \frac{2 \times 10^{-7} I}{n^2} \Big\{ & \frac{1}{4} + \ln\frac{1}{r_1} + \ln\frac{1}{D_{12}} + \ln\frac{1}{D_{13}} + \cdots + \ln\frac{1}{D_{1N}} \\ & + \frac{1}{4} + \ln\frac{1}{D_{21}} + \ln\frac{1}{r_2} + \ln\frac{1}{D_{23}} + \cdots + \ln\frac{1}{D_{2N}} \\ & + \frac{1}{4} + \ln\frac{1}{D_{31}} + \ln\frac{1}{D_{32}} + \ln\frac{1}{r_3} + \cdots + \ln\frac{1}{D_{3N}} \\ & + \cdots\cdots\cdots\cdots\cdots\cdots \\ & + \frac{1}{4} + \ln\frac{1}{D_{N1}} + \ln\frac{1}{D_{N2}} + \ln\frac{1}{D_{N3}} + \cdots + \ln\frac{1}{r_N} \\ & - \ln\frac{1}{D_{1a}} - \ln\frac{1}{D_{1b}} - \ln\frac{1}{D_{1c}} - \cdots - \ln\frac{1}{D_{1n}} \\ & - \ln\frac{1}{D_{2a}} - \ln\frac{1}{D_{2b}} - \ln\frac{1}{D_{2c}} - \cdots - \ln\frac{1}{D_{2n}} \\ & - \ln\frac{1}{D_{3a}} - \ln\frac{1}{D_{3b}} - \ln\frac{1}{D_{3c}} - \cdots - \ln\frac{1}{D_{3n}} \\ & - \cdots\cdots\cdots\cdots\cdots\cdots \\ & - \ln\frac{1}{D_{Na}} - \ln\frac{1}{D_{Nb}} - \ln\frac{1}{D_{Nc}} - \cdots - \ln\frac{1}{D_{Nn}} \Big\} \end{aligned}$$

linkages per m. (19)

Combining the various logarithmic terms, we may write

$$\lambda = 2 \times 10^{-7} I \left\{ \frac{1}{4n} + \ln \frac{\sqrt[n^2]{\begin{matrix} D_{1a}D_{1b}D_{1c} \cdots D_{1n} \cdot D_{2a}D_{2b}D_{2c} \cdots D_{2n} \cdot D_{3a}D_{3b}D_{3c} \\ \cdots D_{3n} \cdots D_{Na}D_{Nb}D_{Nc} \cdots D_{Nn} \end{matrix}}}{\sqrt[n^2]{\begin{matrix} r_1D_{12}D_{13} \cdots D_{1N} \cdot D_{21}r_2D_{23} \cdots D_{2N} \cdot D_{31}D_{32}r_3 \\ \cdots D_{3N} \cdots D_{N1}D_{N2}D_{N3} \cdots r_N \end{matrix}}} \right\}$$

linkages per m. (20)

Now let the number of filaments n into which each conductor is divided approach infinity. The term $1/4n$ obviously approaches zero. The numerator of the term under the logarithm sign approaches the geometric mean distance from one conductor to the other, and the denominator approaches the geometric mean distance from the left-hand conductor to itself. The mutual geometric mean distances (gmd's) will be represented by the symbol D_m, and the self gmd's by the symbol D_s.

We may then write, as the expression for the number of linkages λ about a conductor whose self gmd is D_s and with respect to which the return current has a mutual gmd D_m,

$$\lambda = 2 \times 10^{-7} \, I \ln \frac{D_m}{D_s} \text{ linkages per m,} \tag{21}$$

or

$$\lambda = 0.000\,741\,1 \, I \log_{10} \frac{D_m}{D_s} \text{ linkages per mile.} \tag{22}$$

Dividing these expressions by I gives the inductance in henries.

17. GMD of Various Shapes. The following well-established theorems are useful in computing the inductance of parallel conductors of common designs.

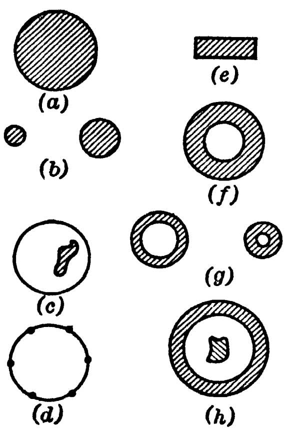


FIG. 9. Various sections whose geometric mean distances are given.

a. The self gmd of a circular area (figure 9*a*) of radius r is equal to

$$D_s = r\epsilon^{-1/4} = 0.7788 \cdots r. \tag{23}$$

b. The gmd between two circular areas external to each other and in a common plane (figure 9*b*) is equal to the distance between their centers. The circles may be of the same or different sizes.

$$D_m = D. \tag{24}$$

c. The gmd from a circular line of radius R to any point, line or area in the same plane and wholly enclosed by the circular line (figure 9*c*) is equal to its radius.

$$D_m = R. \tag{25}$$

d. If a circular line of radius R has around its periphery n equally spaced points (figure 9*d*), the gmd among them is equal to

$$D_m = R \sqrt[n-1]{n}. \tag{26}$$

This is Guye's theorem.

e. The self gmd of a rectangular area (figure 9*e*) whose sides are a and b is very nearly equal to

$$D_s = 0.2235 \, (a + b) \tag{27}$$

for all ratios of a to b. Actually the numerical part of this expression ranges between 0.2231 · · · and 0.2237 · · · for the different possible shapes of rectangle.

f. The self gmd of an annular area of outside and inside radii r_1 and r_2, respectively (figure 9*f*), has as its natural logarithm the following expression:

$$\ln D_s = \ln r_1 - \frac{\dfrac{r_1^4}{4} - r_1^2 r_2^2 + r_2^4\left(\dfrac{3}{4} + \ln\dfrac{r_1}{r_2}\right)}{(r_1^2 - r_2^2)^2}. \tag{28}$$

A working chart for the determination of D_s for an annular section is presented in figure 10.

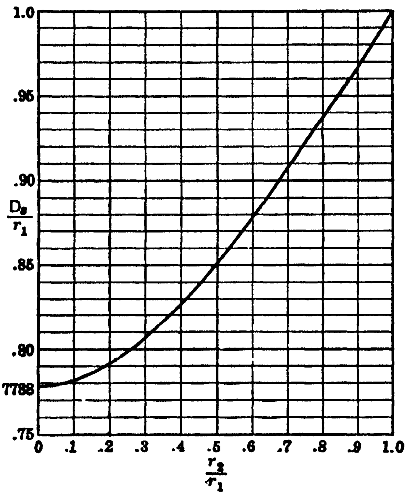


FIG. 10. Chart for determination of D_s for an annular area of outside radius r_1 and inside radius r_2.

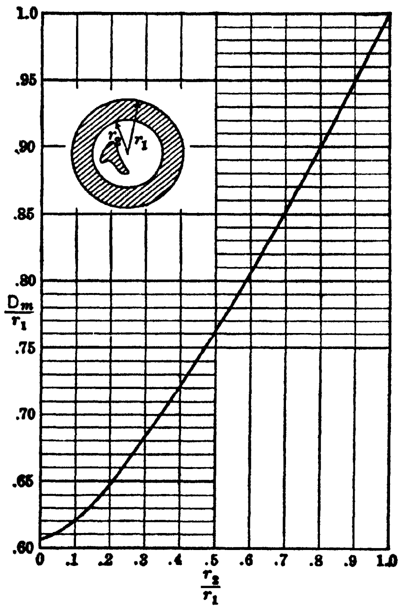


FIG. 11. Chart for determination of D_m from an annular area to another area within the annulus.

g. The gmd between two annular areas in the same plane (figure 9*g*) is equal to the distance between their centers.

$$D_m = D. \tag{29}$$

The two areas may be of different size, and either or both may assume the limiting shapes of circular lines or of full circular areas, or even mere points.

h. The gmd of any point, line or area wholly within an annular area (figure 9*h*) whose outer and inner radii are respectively r_1 and r_2 has for its natural logarithm the following expression:

$$\ln D_m = \frac{r_1^2 \ln r_1 - r_2^2 \ln r_2}{r_1^2 - r_2^2} - \frac{1}{2}. \tag{30}$$

In figure 11 is presented a chart for the graphical determination of this D_m.

18. Values of Self GMD of Common Types of Conductors. In the majority of cases, transmission conductors will be either solid round wires or else cables with concentric stranding, having all strands the same size. If there is a single layer of strands around a central straight strand, 6 strands will be required to fill the annular space. A second layer requires 12 strands more; a third layer 18 more; and so on up in multiples of 6. The total number of strands in such cables is thus 7, 19, 37, 61, 91, 127 or 169. Successive layers are spiraled in opposite directions, so there is no tendency for one layer to settle into the interstices of the one underneath. Since the sizes of such conductors are generally expressed in terms of their areas in circular measure, it is convenient to express their self gmd also in the same terms. Let A represent the area of a cable in circular measure.

1. *Solid round wire.* From (23) the value of self gmd is seen to be $0.7788 \cdots r$. The area A in circular measure is equal to $4\, r^2$, so we have

$$d_s = 0.3894 \cdots \sqrt{A}. \tag{31}$$

The lower-case symbol d_s will be used to designate the self gmd of a single conductor, and the capital-letter symbol D_s to designate the self gmd of an entire phase or group of conductors in parallel. In single-circuit lines there is, of course, no difference.

2. *Concentric cable of* 7 *strands.* Referring to figure 12, in which ρ is the radius of each strand, it is seen that the total area A in circular measure is equal to $7 \times 4\, \rho^2 = 28\, \rho^2$. Assuming uniform distribution of current over the cross section, there are seven equal currents in the seven strands, and the self gmd of the entire current is the forty-ninth root of the product of the 49 individual gmd's among the seven strands, as follows:

$$\begin{aligned} d_s &= \sqrt[49]{(2\,\rho\, \sqrt[5]{6})^{6\times 5}\, (2\,\rho)^{6\times 2}\, (\rho\epsilon^{-1/4})^7} \\ &= 2\,\rho\, \sqrt[49]{3^6\, \epsilon^{-7/4}/2} \\ &= 0.4114 \cdots \sqrt{A}. \end{aligned} \tag{32}$$

In the large radical of (32), the term in the first parenthesis is the gmd among the six outer strands, as given in (26). Mutual distances between pairs of these strands are equal to corresponding mutual distances between their centers, in accordance with (24). There are 6×5 distances, and the product of them all is equal to the thirtieth power of their gmd, as written. The term inside the second parenthesis is the distance from an outer to the inner strand. There are 12 rather than 6 such distances, because both directions must be taken. The last parenthetic expression is the self gmd of each of the seven strands.

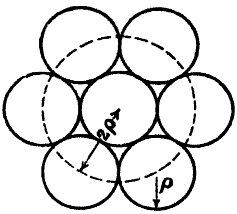


FIG. 12. Cross section of a 7-strand cable.

3. *Concentric cable of* 19 *strands.* For the 19-strand cable, figure 13, the area A in circular mils is equal to $76\,\rho^2$. The value of d_s is

$$d_s = \sqrt[19^2]{\{(4\,\rho\,\sqrt[11]{12})^{12\times 11}\,(2\,\rho\,\sqrt[5]{6})^{6\times 5}\,(4\,\rho)^{12\times 7\times 2}\,(2\,\rho)^{6\times 2}\,(\rho\epsilon^{-1/4})^{19}\}}$$
$$= 0.4345\cdots\sqrt{A}. \qquad (33)$$

The first two parentheses in (33) contain the self gmd's among strands of the outer and second layers respectively. The next term $4\,\rho$ is the gmd from the outer layer to all strands within it, in accordance with (25). The outer layer may be considered as a mere circle because its individual strands may be considered as their center points, and these are spiraled about the center of the cable. The fourth parenthesis contains the gmd of the six strands of the second layer to the center strand. The final term under the radical represents the self gmd of the 19 individual strands.

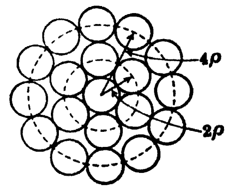


FIG. 13. Cross section of a 19-strand cable.

4. *Concentric cable of* 37 *strands.* The value of self gmd is

$$d_s = \sqrt[37^2]{\{(6\,\rho\,\sqrt[17]{18})^{18\times 17}\,(4\,\rho\,\sqrt[11]{12})^{12\times 11}\,(2\,\rho\,\sqrt[5]{6})^{6\times 5}\,(6\,\rho)^{18\times 19\times 2}\,(4\,\rho)^{12\times 7\times 2}\,(2\,\rho)^{6\times 2}\,(\rho\epsilon^{-1/4})^{37}\}}$$
$$= 0.4418\cdots\sqrt{A}. \qquad (34)$$

5. *Concentric cable of* 61 *strands.* The value of self gmd is

$$d_s = \sqrt[61^2]{\{(8\,\rho\,\sqrt[23]{24})^{24\times 23}\,(6\,\rho\,\sqrt[17]{18})^{18\times 17}(4\,\rho\,\sqrt[11]{12})^{12\times 11}\,(2\,\rho\,\sqrt[5]{6})^{6\times 5}\,(8\,\rho)^{24\times 37\times 2}\,(6\,\rho)^{18\times 19\times 2}\,(4\,\rho)^{12\times 7\times 2}\,(2\,\rho)^{6\times 2}\,(\rho\epsilon^{-1/4})^{61}\}}$$
$$= 0.4448\cdots\sqrt{A}. \qquad (35)$$

6. *Concentric cables of* 91, 127 *and* 169 *strands.* Similarly, the values of d_s for larger cables are: for 91 strands,

$$d_s = 0.4464\cdots\sqrt{A}; \qquad (36)$$

for 127 strands,

$$d_s = 0.4473 \cdots \sqrt{A}; \qquad (37)$$

and for 169 strands,

$$d_s = 0.4478 \cdots \sqrt{A}. \qquad (38)$$

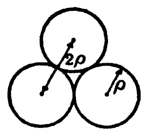


FIG. 14. Cross section of a 3-strand cable.

7. *Cable of three strands.* The form of the three-strand cable is shown in figure 14. There is no central strand, so it should not be considered as a concentric cable. The value of d_s is obviously

$$d_s = \sqrt[9]{(2\,\rho)^6(\rho\epsilon^{-1/4})^3} = 0.421 \cdots \sqrt{A}. \qquad (39)$$

19. Self GMD of Hollow Stranded Conductors. Hollow conductors have been developed principally for the purpose of eliminating or reducing corona formation, but also to reduce skin effect and inductance. A cross section of such a conductor is shown in figure 15. The core is a small twisted copper I-beam. Around this is wrapped, in the opposite direction of spiraling, an inner layer of 18 copper strands. An outer layer of 24 strands is spiraled about the inner layer, the direction of twist again being reversed. This arrangement in used in Anaconda designs 96, 561 and 379.

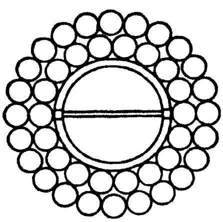


FIG. 15. Anaconda hollow conductor with twisted copper I-beam core surrounded by two layers of strands.

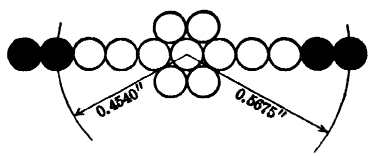


FIG. 16. Working drawing for computation of self gmd of Anaconda design 316.

A working drawing for the computation of the self gmd of Anaconda design 316 (see table VI) is shown in figure 16. There are 24 and 30 strands in the two layers. The area of the 54 strands is equal to $54\,(0.1135)^2\,\frac{\pi}{4} = 0.5462$ square inch, and the total area including core is given as 0.6057 square inch. The area of the core is therefore 0.0595 square inch. The two layers of strands, 24 and 30, correspond to what would be the fourth and fifth layers in an ordinary concentric cable, as shown in figure 16. The inside diameter of the hollow space is equal to the diameter of 7 strands, or 0.7945 inch. On the assumption that the core piece is a rectangle, its dimensions would be 0.7945×0.075 inch, and its area of 0.0595 is equal to that of 5.89 strands. The total area is the equivalent of 59.89 strands. The value of d_s is given by the following expression:

$$d_s = \sqrt[59.89^2]{\{(0.5675 \sqrt[29]{30})^{29\times 30} (0.4540 \sqrt[23]{24})^{23\times 24} (0.5675)^{30\times 29.89\times 2} (0.4540)^{24\times 5.89\times 2} [0.2235(0.7945 + 0.075)]^{5.89^2} (0.7788 \times 0.05675)^{54}\}} = 0.540 \text{ in.} \quad (40)$$

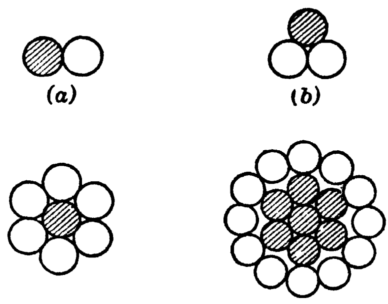
The six terms under the radical are, in order: (1) the gmd among the outer layer; (2) among the inner layer; (3) from outer layer to inner layer and core, and return; (4) from inner layer to core, and return; (5) from core to itself; and (6) from the strands to themselves.

This computation indicates a d_s of 0.540 inch, whereas the tabulated value of 0.555, table VI, was obtained by working backward from the Anaconda inductance tables. The difference of 2.7 per cent in gmd will affect the actual value of inductance by less than 0.5 per cent, and this difference may be accounted for by the reduction in inductance due to skin effect at 60 cps. It will be noted that a *reduction* in inductance is accompanied by an *increase* in equivalent self gmd, other things being equal.

It is doubtful if we should expect a precision much better than 0.5 per cent in inductance computations in view of the possible variations in lay of spiraling, strand size, loosening, sag and other factors, many of which are discussed in one of the references.

20. Non-Homogeneous Conductors. The fundamental inductance formula as given in terms of gmd is capable of being applied to the more general case of non-homogeneous distribution of current in the conductors, provided that the gmd taken is that of one current cross section to another or to itself, and not strictly that between areas, unless indeed the current is uniformly distributed over these areas.

Composite conductors of copper and bronze are in fairly common use. For these, as stated previously, the gmd to be used depends on the relative conductivity of the two metals, and not merely on the geometry of the cross section. The four arrangements shown in figure 17 will be considered, and their values of self gmd worked out and tabulated. The unshaded circles will be taken to represent strands of



(a) (b) (c) (d)

FIG. 17. Composite stranded conductors of copper and bronze.

unit conductivity, and the cross-hatched circles will represent those of conductivitv γ (relative to the others). The relative values of current density in the two materials are assumed to be directly proportional to the conductivities. This is tantamount only to neglecting skin effect and proximity effect. In building up the gmd's of the entire conductors from those of the individual strands, it is necessary to weight the various component distances in accordance with the currents in the strands. The distance between two strands (or from one strand to itself) carrying *unit* current is weighted as 1×1 or 1; between two strands of which one carries unity and the other γ current, as $1 \times \gamma$ or γ; between two strands (or one strand to itself) carrying γ current, as $\gamma \times \gamma$ or γ^2. For strands which carry different currents because of having different sizes, the same sort of weighting must be applied.

For two strands of equal radius but different non-magnetic materials (e.g., one copper and the other bronze) as illustrated in figure 17*a*,

$$d_s = \rho \sqrt[(1+\gamma)^2]{2^{2\gamma} (0.7788)^{1+\gamma^2}}. \tag{41}$$

If there are three equal strands, as shown in figure 17*b*, and two strands have conductivity unity and the remaining strand conductivity γ,

$$d_s = \rho \cdot \sqrt[(2+\gamma)^2]{4^{1+2\gamma} (0.7788)^{2+\gamma^2}}. \tag{42}$$

If there are seven equal strands, as shown in figure 17*c*, with the six outer ones of unit conductivity and the central one of conductivity γ,

$$\begin{aligned} d_s &= \sqrt[(6+\gamma)^2]{(2\rho \sqrt[5]{6})^{30} (2\rho)^{12\gamma} (0.7788\,\rho)^{6+\gamma^2}} \\ &= \rho \cdot \sqrt[(6+\gamma)^2]{2^{36+12\gamma}\, 3^6 (0.7788)^{6+\gamma^2}}. \end{aligned} \tag{43}$$

In (43), the gmd from the outer six strands to one another is equal by (26) to $2\rho \sqrt[5]{6}$; the number of individual distances among these strands is $6 \times 5 = 30$, and the weighting of each distance is unity. This gives the expression $(2\rho \sqrt[5]{6})^{30}$ under the first radical. The distance from each outer strand to the central strand is 2ρ, and there are 6 such distances as well as 6 more coming back; total 12. The weighting is $1 \times \gamma$. Thus the term $(2\rho)^{12\gamma}$ is arrived at. The gmd of each strand to itself is $\rho\epsilon^{-1/4} = 0.7788\,\rho$. Six of these distances have to be weighted as unity, and the seventh as γ^2, giving in all $(0.7788\,\rho)^{6+\gamma^2}$.

In figure 17*d* is shown a conductor having 19 equal strands. Let the inner 7 have conductivity γ and the outer 12 conductivity unity.

Then we may write:

$$d_s = \sqrt[(12+7\gamma)^2]{\{(4\,\rho\sqrt[11]{12})^{11\times12}\,(4\,\rho)^{12\times7\gamma\times2}\,(2\,\rho\sqrt[5]{6})^{6\gamma\times5\gamma}\,(2\,\rho)^{6\gamma\times\gamma\times2}\,(0.7788\,\rho)^{12+7\gamma^2}\}}. \tag{44}$$

The first parenthetical expression under this radical is the gmd of the 11 $\times$ 12 distances among the outer strands (weighted unity); the second, the gmd from the 12 outer to the 7 inner strands and return (weighted γ); the third, the gmd of the 5 $\times$ 6 distances among the strands of the second layer (weighted γ^2); the fourth, from the second layer to the central strand (weighted γ^2); and the fifth and last, the gmd of each strand to itself, 12 being given unit weighting and 7 being weighted γ^2. The expression can be somewhat simplified to the form:

$$d_s = \rho \cdot \sqrt[(12+7\gamma)^2]{2^{288+336\gamma+48\gamma^2}\,3^{12+6\gamma^2}\,(0.7788)^{12+7\gamma^2}}. \tag{45}$$

Example. A copper-bronze composite cable (Anaconda design 60) has a center strand of " Calsun " bronze of 15 per cent conductivity, and around it six copper strands of unit conductivity. The diameter of each strand is 0.1166 inch. Find the self gmd of this conductor.

Solution.

$$d_s = 0.0583\,\sqrt[6.15^2]{2^{37.8}\,3^6\,0.7788^{6.022}}$$
$$= 0.0583\,(2^{1.000}\,3^{0.1588}\,0.7788^{0.1593})$$
$$= 0.0583 \times 2 \times 1.191 \div 1.041 = 0.1333 \text{ inch.}$$

Example. As a further example consider a single-phase circuit, as shown in figure 18, in which each conductor is composed of two non-magnetic twisted strands, each strand of radius ρ; we designate the distance on centers of the two conductors as D. Consider two cases: (1) in which all strands are composed of the same material, and (2) in which one strand in each conductor has unit conductivity and the other strand conductivity γ.

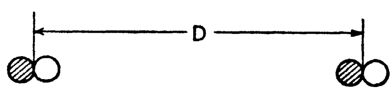


FIG. 18. Return circuit of two-stranded composite conductors.

Case 1. The self gmd of a complete conductor is $\sqrt{2\,\rho \cdot \rho\epsilon^{-1/4}} = \rho\,\sqrt{1.558} = 1.248\,\rho$ The inductance per conductor is

$$L = 2 \times 10^{-7} \ln \frac{D}{1.248\,\rho} \text{ h per m.}$$

Case 2. The self gmd of a complete conductor is

$$\sqrt[(1+\gamma)^2]{(2\,\rho)^{2\gamma}\,(\rho\epsilon^{-1/4})^{1+\gamma^2}} = \rho \cdot \sqrt[(1+\gamma)^2]{2^{2\gamma}\,(\epsilon^{-1/4})^{1+\gamma^2}}.$$

The inductance per conductor is:

$$L = 2 \times 10^{-7} \ln \frac{D}{\rho \cdot \sqrt[(1+\gamma)^2]{2^{2\gamma}\,(\epsilon^{-1/4})^{1+\gamma^2}}} \text{ h per m.}$$

For the special case where $\gamma = 1$, this formula simplifies to the expression of case 1.

21. Equivalent Self GMD. The use of the gmd in the computation of inductance is usually considered to require uniform distribution of current over the cross section of the conductor, and also the absence of any magnetic material in the vicinity. It has already been seen that, for certain types of non-uniform current distribution, the method may still be applied, provided the gmd of the *current* cross section rather than *area* cross section is used.

It is convenient to make a further extension of the method, to cover certain cases in which magnetic materials are present, or where there is appreciable skin effect, or both. This is done by introducing a term which will be called *equivalent self gmd.* Let this hypothetical distance be the one which, if inserted for D_s in the fundamental inductance formula

$$L = 2 \times 10^{-7} \ln \frac{D_m}{D_s} \text{ h per m,} \tag{46}$$

will give the true inductance. The true value of the inductance may be determined for example by experiment or by a series of computations in which various minor factors, such as spiraling and skin effect, are taken into account. It may be determined by a combination of calculation and experiment, such as has been used for the determination of the inductance of steel-reenforced aluminum cable.

In general, d_s is the term in the inductance equation which brings in the effect of the so-called internal inductance of the conductor. It makes no difference in the total inductance of a circuit, or in the linkages with distant conductors, whether a given value of d_s is an actual self gmd, or whether it is a mere hypothetical distance which has been largely affected by magnetic material in the conductor.

Since the purpose of finding the self gmd is ordinarily to aid in the subsequent determination of inductance, it may seem a useless and foolish step to work back from a known inductance to a value of equivalent self gmd. The purpose, however, is to obtain, in whatever way, the true equivalent self gmd in order to use it later to find the inductance of multicircuit lines, and for different arrangements and spacings of single-circuit lines.

The value of equivalent self gmd may be expressed explicitly in terms of inductance and mutual gmd, as follows:

$$D_s = D_m \epsilon^{-10^7 L/2}, \tag{47}$$

in which L is expressed in henries per meter. In terms of 60-cps reactance, x_{60}, in ohms per mile,

$$D_s = D_m \, 10^{-x_{60}/0.2794}. \tag{48}$$

22. Inductance of Steel-Reenforced Aluminum Cable. Consider the cable whose cross section is shown in figure 19, and which has 54 aluminum and 7 steel strands. This is known as a "54 + 7 strand A.C.S.R." The major part of the current flows in the aluminum, both because of its larger area and its greater conductivity. The inductance for this part alone (of course, with an appropriate return

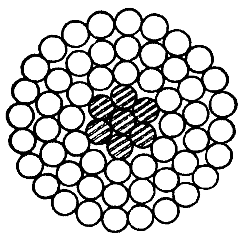


FIG. 19. A.C.S.R. with 54 aluminum and 7 steel strands.

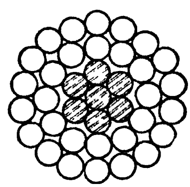


FIG. 20. A.C.S.R. with 30 aluminum and 7 steel strands.

path) may be computed, and the effect of the steel reenforcement treated as a small correction to be applied later.

The self gmd of the 54 aluminum strands alone is quite similar to (35), which is for the 61-strand concentric cable. For the 54 strands,

$$\begin{aligned} d_s &= \sqrt[54^2]{\{(8\rho \sqrt[23]{24})^{24\times 23} (6\rho \sqrt[17]{18})^{18\times 17} (4\rho \sqrt[11]{12})^{12\times 11} (8\rho)^{24\times 30\times 2} (6\rho)^{18\times 12\times 2} (\rho\epsilon^{-1/4})^{54}\}} \\ &= 7.289\,\rho \\ &= 0.4959\sqrt{A}, \end{aligned} \tag{49}$$

where A is the total area of the aluminum expressed in circular measure, equal to $216\,\rho^2$.

For the 30 + 7 strand A.C.S.R., figure 20, the corresponding value of self gmd is

$$\begin{aligned} d_s &= \sqrt[30^2]{\{(6\rho \sqrt[17]{18})^{18\times 17} (4\rho \sqrt[11]{12})^{12\times 11} (6\rho)^{18\times 12\times 2} (\rho\epsilon^{-1/4})^{30}\}} \\ &= 5.784\,\rho \\ &= 0.5280\sqrt{A}, \end{aligned} \tag{50}$$

where A is equal to $120\,\rho^2$.

The effect of the steel core will be to increase the number of linkages over what it would be if the cable were merely hollow, both because of linkages in the core itself, and because the shifting of part of the current toward the center results in greater flux density throughout the aluminum portion.

The particular case of a 556,500-circular-mil 30 + 7 strand cable at a frequency of 60 cps will be considered. Skin effect is very small and will be neglected. The resistivity of the steel is about 110 ohms per mil foot at 20 C, and of aluminum about 17.0 ohms per mil foot. The aluminum will carry $\frac{30}{7} \times \frac{110}{17.0} = 27.7$ times as much current as the steel. The steel will carry 1/28.7 or 3.48 per cent in the cable under consideration. We will compute the effect of this core current on the inductance and apply it as a correction to the calculations already made, in which the effect of core current was neglected. First we note that the removal of 3.48 per cent of the current from the aluminum and its flow in the steel has no effect whatever upon the magnetic flux external to the entire cable. It will increase the flux density in the aluminum region by an amount which varies from zero at the outside to a maximum at the inside of the aluminum, where before the flux density was zero. The flux density in the steel is increased also, the value on the basis of assumed zero core current being zero throughout. If now the aluminum be considered as a tube of inside diameter 0.4086 inch and outside diameter 0.9534 inch, the increase in flux linkages per meter length due to the increased flux in the aluminum is equal to

$$10^{-7}\left\{\int_{0.2043}^{0.4767} \frac{4\,\pi I}{2\,\pi x}\left[0.0348 + 0.9652\,\frac{x^2 - 0.2043^2}{0.4767^2 - 0.2043^2}\right]^2 dx \right.$$

$$\left. - \int_{0.2043}^{0.4767} \frac{4\,\pi I}{2\,\pi x}\left[\frac{x^2 - 0.2043^2}{0.4767^2 - 0.2043^2}\right]^2 dx\right\}$$

$$= 10^{-7}\,(0.378\,I - 0.358\,I) = 2.0 \times 10^{-9}\,I \text{ linkage per m,}$$

where I is expressed in amperes. Applying the proper conversion factors we find that this represents an increase in inductance per mile of one conductor amounting to 0.00322 millihenry.

In addition to this we have another increase in inductance, owing to the linkages between the core current and core flux. Experimental results on 7-strand steel cables of the size of this core indicate an internal inductance which varies slightly with the current, but an average value is 1.50 mh per mile. In order to make use of this datum for the A.C.S.R., it must be multiplied by the square of the fraction of current carried by the steel, or 0.0348. The contribution to the inductance is therefore $0.0348^2 \times 1.50 = 0.00182$ mh per mile. The total increase due to the presence of the steel in the center instead of a non-conductor is $0.00322 + 0.00182 = 0.00504$ mh per mile. The same increase would apply to other A.C.S.R. of different sizes provided they are geometrically similar, i.e., composed of 30 + 7 strands of aluminum and steel respectively.

For the 54 + 7 strand A.C.S.R. the corresponding increase works out to be $0.0017 + 0.0006 = 0.0023$ mh per mile.

These corrections, added to the inductance based on the aluminum alone, form the basis for the equivalent self gmd figures presented in table V.

23. Tables of Self GMD and Resistance. Considerable saving may be effected in time, effort, and probability of error by using carefully checked tables of gmd and of reactance, rather than computing each case in its entirety. A number of working tables have been prepared and are presented for these reasons.

Table IV gives values of self gmd for solid and concentric stranded conductors of the range of sizes used in transmission. The temperature coefficient of resistance is 0.00385 per degree at 25 C. The resistance figures in the table are for standard annealed copper of 100 per cent conductivity. For hard-drawn copper the resistance is 2.7 per cent greater.

The values of d_s given in table IV would apply also to aluminum conductors, but the conductivity of aluminum is only 61 per cent of that of standard annealed copper. All-aluminum conductors, however, are seldom used on account of mechanical difficulties, which are overcome largely by the use of steel reenforcement.

Table V gives values of self gmd and resistance for a number of standard designs of steel-reenforced aluminum cable.

Table VI gives some of the dimensions and the values of self gmd for Anaconda hollow conductors.

TABLE IV

VALUES OF SELF GMD AND RESISTANCE FOR SOLID AND CONCENTRIC STRANDED CONDUCTORS

Resistance Based on Annealed Copper of 100% Conductivity

B & S Gauge	Circular Mils	Solid Wire			Stranded Concentric Cable					
		d_s (inches)	Resistance in Ohms per Mile at 25 C		Standard		Resistance in Ohms per Mile at 25 C		Flexible	
			d-c	60-cps	Strands	d_s (inches)	d-c	60-cps	Strands	d_s (inches)
	2,000,000				127	0.6326	0.0285	0.0371	169	0.6333
	1,900,000				127	0.6166	0.0300	0.0383	169	0.6172
	1,800,000				127	0.6002	0.0316	0.0397	169	0.6008
	1,700,000				127	0.5833	0.0335	0.0413	169	0.5838
	1,600,000				127	0.5658	0.0356	0.0431	169	0.5664
	1,500,000				91	0.5468	0.0380	0.0451	127	0.5479
	1,400,000				91	0.5282	0.0407	0.0475	127	0.5293
	1,300,000				91	0.5090	0.0438	0.0502	127	0.5100
	1,200,000				91	0.4890	0.0475	0.0535	127	0.4900
	1,100,000				91	0.4703	0.0518	0.0574	127	0.4713
	1,000,000				61	0.4448	0.0570	0.0622	91	0.4464
	950,000				61	0.4336	0.0600	0.0649	91	0.4351
	900,000				61	0.4220	0.0633	0.0680	91	0.4235
	850,000				61	0.4101	0.0670	0.0715	91	0.4116
	800,000				61	0.3979	0.0712	0.0755	91	0.3993
	750,000				61	0.3853	0.0759	0.0799	91	0.3866
	700,000				61	0.3722	0.0814	0.0851	91	0.3735
	650,000				61	0.3586	0.0876	0.0911	91	0.3599
	600,000				61	0.3446	0.0949	0.0982	91	0.3458
	550,000				61	0.3299	0.1035	0.1065	91	0.3311

TABLE IV — (*Continued*)

	500,000				37	0 3124	0 1139	0 1166	61	0 3146
	450,000				37	0 2964	0 1266	0 1290	61	0 2984
	400,000				37	0 2794	0 1424	0 1446	61	0 2813
	350,000				37	0 2614	0 1627	0 1646	61	0 2632
	300,000				37	0 2420	0 1899	0 1915	61	0 2437
	250,000				37	0 2209	0 2278	0 2292	61	0 2224
0000	211,600	0 1791	0 264	0 265	19	0 1999	0 2691	0 2703	37	0 2032
000	167,800	0 1595	0 333	0 334	19	0 1780	0 339	0 340	37	0 1810
00	133,100	0 1421	0 420	0 421	19	0 1585	0 428	0 429	37	0 1612
0	105,500	0 1265	0 528	0 529	19	0 1412	0 540	0 540	37	0 1435
1	83,700	0 1127	0 665	0 665	19	0 1257	0 681	0 681	37	0 1278
2	66,400	0 1003	0 840	0 840	7	0 1060	0 858	0 858	19	0 1119
3	52,600	0 0893	1 06	1 06	7	0 0944	1 082	1 082	19	0 0997
4	41,700	0 0796	1 34	1 34	7	0 0840	1 365	1 365	19	0 0888
5	33,100	0 0708	1 68	1 68	7	0 0748	1 721	1 721	19	0 0791
6	26,250	0 0631	2 13	2 13	7	0 0667	2 170	2 170	19	0 0704
7	20,820	0 0562	2 68	2 68	7	0 0594	2 74	2 74	19	0 0627
8	16,510	0 0500	3 38	3 38	7	0 0529	3 45	3 45	19	0 0558
9	13,090	0 0446	4 27	4 27						
10	10,380	0 0397	5 39	5 39						
11	8,230	0 0353	6 76	6 76						
12	6,530	0 0315	8 55	8 55						
13	5,180	0 0280	10 8	10 8						
14	4,110	0 0250	13 6	13 6						
15	3,260	0 0222	17 2	17 2						
16	2,580	0 0198	21 6	21 6						

TABLE V

VALUES OF SELF GMD AND RESISTANCE FOR STEEL-REENFORCED ALUMINUM CONDUCTORS

Circular Mils Aluminum	No. of Strands		d_s (inches)	Resistance in Ohms per Mile at 25 C	
	Aluminum	Steel		d-c	60-cps 200 Amp per sq in.
1,590,000	54	7	0.619	0.0587	0.0594
1,510,500	54	7	0.603	0.0618	0.0625
1,431,000	54	7	0.585	0.0652	0.0659
1,351,500	54	7	0.569	0.0691	0.0698
1,272,000	54	7	0.553	0.0734	0.0742
1,192,500	54	7	0.535	0.0783	0.0791
1,113,000	54	7	0.518	0.0839	0.0848
1,033,500	54	7	0.501	0.0903	0.0913
954,000	54	7	0.480	0.0979	0.0985
900,000	54	7	0.465	0.104	0.105
874,500	54	7	0.460	0.107	0.108
795,000	54	7	0.438	0.117	0.119
715,500	54	7	0.414	0.131	0.133
666,600	54	7	0.401	0.140	0.142
636,000	54	7	0.391	0.147	0.149
605,000	54	7	0.382	0.154	0.156
556,500	30	7	0.386	0.168	0.168
500,000	30	7	0.365	0.187	0.187
477,000	30	7	0.358	0.196	0.196
397,500	30	7	0.328	0.235	0.235
336,400	30	7	0.298	0.278	0.278
300,000	30	7	0.290	0.311	0.311
1,590,000	54	19	0.625	0.0587	0.0594
1,510,500	54	19	0.611	0.0618	0.0625
1,431,000	54	19	0.592	0.0652	0.0659
1,351,500	54	19	0.577	0.0691	0.0698
1,272,000	54	19	0.557	0.0734	0.0742
1,192,500	54	19	0.540	0.0783	0.0791
1,113,000	54	19	0.521	0.0839	0.0848
795,000	30	19	0.473	0.117	0.117
715,500	30	19	0.447	0.131	0.131
636,000	30	19	0.421	0.147	0.147
300,000	26	7	0.274	0.311	0.311
266,800	26	7	0.258	0.350	0.350
266,800	6	7	0.237	0.350	0.352
211,600	6	1	0.145	0.441	0.446
167,800	6	1	0.131	0.556	0.560
133,100	6	1	0.120	0.702	0.707
105,500	6	1	0.107	0.885	0.889
83,700	6	1	0.0976	1.12	1.12
66,400	6	1	0.0889	1.41	1.41

TABLE VI

VALUES OF SELF GMD FOR ANACONDA HOLLOW CONDUCTORS

Design Number	Equivalent Hand-Drawn Copper Conductance	Outside Diameter (inches)	Area (sq in.)	No. of Layers	Strands		d_s (inches)
					Number	Diameter (inches)	
163	800,000	1.269	0.6426	2	22 + 28	0.1228	0.560
167	800,000	1.185	0.6380	2	16 + 22	0.1422	0.509
316	750,000	1.249	0.6057	2	24 + 30	0.1135	0.555
96	750,000	1.176	0.5999	2	18 + 24	0.1307	0.511
362	750,000	1.092	0.5959	2	12 + 18	0.1560	0.456
164	700,000	1.230	0.5664	2	26 + 32	0.1054	0.552
559	700,000	1.171	0.5629	2	21 + 27	0.1171	0.515
361	700,000	1.100	0.5566	2	16 + 20	0.1284–0.1435	0.468
488	650,000	1.252	0.5271	2	32 + 38	0.0916	0.566
560	650,000	1.106	0.5206	2	19 + 25	0.1185	0.484
561	650,000	1.015	0.5165	2	12 + 18	0.1450	0.424
165	600,000	1.245	0.4897	2	36 + 42	0.0830	0.571
360	600,000	1.100	0.4829	2	22 + 28	0.1065	0.485
30	600,000	1.027	0.4781	2	16 + 22	0.1232	0.440
562	550,000	1.148	0.4485	2	32 + 38	0.0840	0.520
563	550,000	1.075	0.4428	2	24 + 30	0.0977	0.499
379	550,000	1.008	0.4399	2	18 + 24	0.1120	0.436
33	500,000	1.079	0.4056	2	30 + 36	0.0830	0.490
4-B	500,000	1.002	0.4039	2	22 + 28	0.0970	0.443
401	500,000	0.995	0.4009	1	12	0.1990	0.436
405	450,000	1.180	0.3737	1	26	0.1221	0.552
564	450,000	1.000	0.3640	2	27 + 33	0.0833	0.449
426	450,000	1.000	0.3645	1	15	0.1667	0.449
51	400,000	1.005	0.3277	1	19	0.1370	0.458
565	350,000	0.968	0.2888	1	21	0.1210	0.445
378	350,000	0.742	0.2796	2	12 + 18	0.1060	0.310
566	350,000	0.720	0.2764	2	10 + 15	0.1108–0.1200	0.295
178	300,000	0.763	0.2413	1	12	0.1525	0.335
38	300,000	0.747	0.2420	1	11	0.1600	0.326
567	300,000	0.710	0.2407	1	9	0.1775	0.303
37	250,000	0.754	0.2023	1	16	0.1190	0.340
415	250,000	0.725	0.2027	1	14	0.1280	0.322
24	250,000	0.695	0.2025	1	12	0.1390	0.306
446	211,600	0.755	0.1734	1	21	0.0944	0.347
448	211,600	0.705	0.1730	1	17	0.1058	0.319
50	211,600	0.665	0.1722	1	14	0.1173	0.295
158	167,800	0.611	0.1372	1	16	0.0964	0.276
569	211,600	0.598	0.1698	1	9	0.1495	0.255
568	211,600	0.583	0.1695	1	8	0.1590	0.246
495	167,800	0.600	0.1376	1	15	0.1000	0.269
570	167,800	0.570	0.1358	1	12	0.1140	0.250
10	133,100	0.550	0.1100	1	17	0.0825	0.249

24. Examples of Inductance Calculation. We have the following formulas for inductance and reactance:

$$L = 2 \times 10^{-7} \ln \frac{D_m}{D_s} \text{ h per m;} \tag{51}$$

$$L = 0.7411 \log_{10} \frac{D_m}{D_s} \text{ mh per mile;} \tag{52}$$

$$x_{60} = 0.2794 \log_{10} \frac{D_m}{D_s} \text{ ohms per mile at 60 cps.} \tag{53}$$

First let us compare equation (8) for a solid round wire with (51) above, assuming that the relative permeability μ/μ_0 of the wire is unity.

$$L = 10^{-7}\left(2 \ln \frac{D}{r} + \frac{1}{2}\right) \tag{8}$$

$$= 10^{-7}\left(2 \ln \frac{D}{r} + \ln \sqrt{\epsilon}\right)$$

$$= 2 \times 10^{-7} \ln \frac{D}{r\epsilon^{-1/4}}. \tag{54}$$

It is seen that the numerator in (54) is the mutual gmd of the conductors of figure 6, and the denominator is the self gmd, in accordance with (23).

Instead of repeatedly looking up the logarithm of the ratio D_m/D_s and then multiplying by a constant to find the reactance, the reactance may be tabulated as a direct function of this ratio. Table VII presents values of 60-cps reactance in ohms per mile as a function of D_m/D_s.

Examples. Find the 60-cps reactance per mile of one conductor of a single-phase line with 120-inch spacing in each case, for each of the following conductors:

(*a*) No. 0000 solid copper.
(*b*) 600,000-cir-mil copper, with standard concentric stranding.
(*c*) Anaconda hollow conductor 164.
(*d*) A.C.S.R., 500,000-cir-mil, 30 + 7 strands.

Solution. The values of self gmd, from the tables, are: (*a*) 0.1791 inch; (*b*) 0.3446 inch; (*c*) 0.552 inch; (*d*) 0.365 inch.

Dividing each of these values of D_s into 120 gives D_m/D_s. The results are: (*a*) 670; (*b*) 348; (*c*) 217; (*d*) 329.

By reference to table VII, the respective reactances are found to be: (*a*) 0.790 ohm per mile; (*b*) 0.710 ohm per mile; (*c*) 0.652 ohm per mile; (*d*) 0.703 ohm per mile.

TABLE VII

60-CPS REACTANCE PER MILE AS A FUNCTION OF D_m/D_s.

D_m/D_s	x_{60} in Ohms per Mile	D_m/D_s	x_{60} in Ohms per Mile	D_m/D_s	x_{60} in Ohms per Mile	D_m/D_s	x_{60} in Ohms per Mile
2500	0.949	1550	0.891	600	0.776	90	0.546
2450	0.947	1500	0.887	550	0.766	80	0.532
2400	0.944	1450	0.883	500	0.754	70	0.516
2350	0.942	1400	0.879	450	0.741	60	0.497
2300	0.939	1350	0.875	400	0.727	50	0.475
2250	0.937	1300	0.870	350	0.711	40	0.448
2200	0.934	1250	0.865	300	0.692	30	0.413
2150	0.931	1200	0.860	250	0.670	25	0.391
2100	0.928	1150	0.855	200	0.643	20	0.364
2050	0.925	1100	0.850	190	0.637	15	0.329
2000	0 922	1050	0.844	180	0.630	10	0.279
1950	0.919	1000	0.838	170	0.623	9	0.267
1900	0.916	950	0.832	160	0.616	8	0.252
1850	0.913	900	0.825	150	0.608	7	0.236
1800	0.910	850	0.819	140	0.600	6	0.218
1750	0.906	800	0.811	130	0.591	5	0.196
1700	0.903	750	0.803	120	0.581	4	0.168
1650	0.899	700	0.795	110	0.570		
1600	0.895	650	0.786	100	0.559		

25. Polyphase Lines. The expression for the number of linkages about any one of a group of n round wires carrying current was developed in article 15. If the wires are not round it is only necessary to replace the self gmd of the round wire by that of the particular type of conductor used. Corresponding to equations (16) and (17) for round wires we may write:

$$\lambda_a = 2 \times 10^{-7} \sum_{j=a}^{n} I_j \ln \frac{1}{D_{ja}} \text{ linkages per m;} \tag{55}$$

$$\lambda_a = 0.000\,741\,1 \sum_{j=a}^{n} I_j \log_{10} \frac{1}{D_{ja}} \text{ linkages per mile.} \tag{56}$$

In these two equations, D_{aa} is to be interpreted as the self gmd of wire a. For the round wire, $\ln 1/d_s$ is equal to $\ln 1/r\epsilon^{-1/4}$, or $1/4 + \ln 1/r$.

The polyphase line is merely a special case of the general n-conductor line and will be so treated.

Equilateral-triangle arrangement. The simplest three-phase arrangement, and the only one which does not inherently require transposition to balance the phase reactance drops, is the equilateral triangle, shown in figure 21. All the interconductor spacings are equal to D, and if the three conductors are designated a, b and c, then the linkages about a (in the absence of zero-sequence current) will be equal to

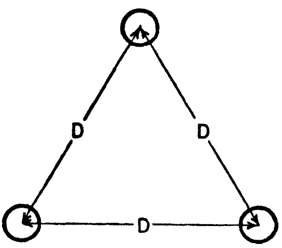


FIG. 21. Cross section of three-phase line with equilateral spacing.

$$\lambda_a = 2 \times 10^{-7}\left(I_a \ln \frac{1}{D_s} + I_b \ln \frac{1}{D} + I_c \ln \frac{1}{D}\right)$$

$$= 2 \times 10^{-7}\, I_a \ln \frac{D}{D_s} \text{ linkages per m.} \qquad (57)$$

Dividing λ_a by I_a gives the inductance L_a.

$$L_a = 2 \times 10^{-7} \ln \frac{D}{D_s} \text{ henry per m;} \qquad (58)$$

or

$$L_a = 0.7411 \log_{10} \frac{D}{D_s} \text{ mh per mile.} \qquad (59)$$

$$x_{60} = 0.2794 \log_{10} \frac{D}{D_s} \text{ ohms per mile at 60-cps.} \qquad (60)$$

It should be noted that the inductance per unit length is the same for a three-phase as for a single-phase line with the same conductor size and spacing.

Unsymmetrical three-phase line. Unsymmetrical lines are in common use for three-wire three-phase transmission, because of more convenient mounting on poles or towers, or for the purpose of keeping the average height above ground of the conductors as low as practicable in order to minimize hazards due to lightning.

If there is no transposition, then the general expression (56) for linkages should be used for each conductor independently. The drop in any wire a will be

$$I_a R_a + j\omega\lambda_a \text{ volts,}$$

applying, of course, to the same distance to which the resistance R_a and linkages λ_a apply. In general, the number of linkages for the different conductors will be different, and under load conditions unbalanced voltages will be produced.

In order to bring the reactances into balance, as well as to minimize inductive interference with parallel communication circuits and to balance the phase charging currents, the conductors should be transposed. Figure 22 represents a cross section of an unsymmetrical line, and figure 23 is the simplest and best transposition cycle to be used in connection with such an arrangement. Let the three conductors

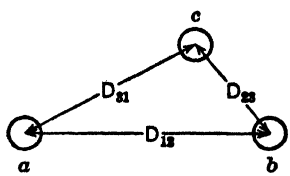


FIG. 22. Cross section of three-phase line with unsymmetrical spacing.

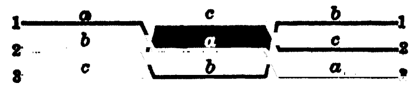


FIG. 23. Transposition cycle for three-wire three-phase line.

be a, b and c; and the three positions 1, 2 and 3. The linkages per meter about a may be computed for each of the three sections, totaled and divided by 3 to get the average. The result is:

$$\lambda_a = 2 \times 10^{-7} I_a \ln \frac{\sqrt[3]{D_{12}D_{23}D_{31}}}{D_s} \text{ linkages per m,}$$

which indicates an inductance of

$$L_a = 2 \times 10^{-7} \ln \frac{\sqrt[3]{D_{12}D_{23}D_{31}}}{D_s} \text{ h per m.} \qquad (61)$$

Another and simpler proof is to approach the problem as in the single-phase case. Observe that the return current of conductor a flows in b and c in combination. Each of these two has a gmd from a, over the transposition cycle, of $\sqrt[3]{D_{12}D_{23}D_{31}}$. The standard gmd formula may be applied:

$$L_a = 2 \times 10^{-7} \ln \frac{D_m}{D_s}$$

$$= 2 \times 10^{-7} \ln \frac{\sqrt[3]{D_{12}D_{23}D_{31}}}{D_s} \text{ h per m.} \qquad (61a)$$

$$L_a = 0.7411 \log_{10} \frac{\sqrt[3]{D_{12}D_{23}D_{31}}}{D_s} \text{ mh per mile} \qquad (61b)$$

$$x_{60} = 0.2794 \log_{10} \frac{\sqrt[3]{D_{12}D_{23}D_{31}}}{D_s} \text{ ohm per mile at 60 cps.} \qquad (61c)$$

26. Multicircuit Lines. The general linkages formula (57) applies to multicircuit lines, both single-phase and polyphase. The line should be transposed so as to afford reactance balance in the different phases, and the current in the different conductors of each phase should be the same if all conductors are of the same design. If this latter condition is not fulfilled, then there will result more ohmic loss than is necessary

Single-phase case. The fundamental formula for the inductance of parallel conductors, such as are shown in figure 8, is

$$L = 2 \times 10^{-7} \ln \frac{D_m}{D_s} \text{ h per m.} \tag{46}$$

There is no limitation as to the shapes of the two sections in this figure. They might each, for example, be divided into two or more non-contiguous portions. Provided conditions still remain such that the distribution of current density is uniform over *all* of each conductor, the formula would still hold good.

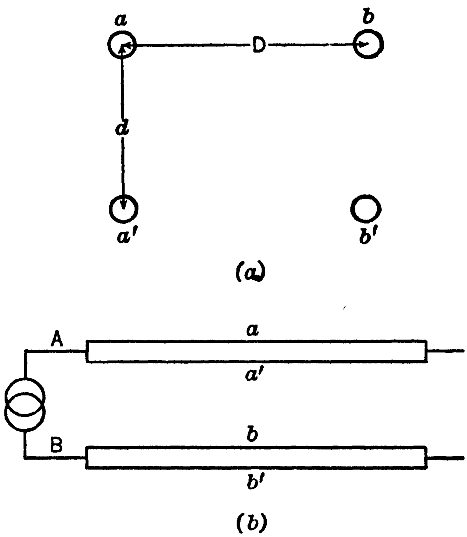


FIG. 24. Double-circuit single-phase line.

Let the configuration be such as is shown in figure 24*a*. Regarding this arrangement as a variant of that of figure 8, conductor A has been metamorphosed into two separate conductors a and a', and B has become two conductors b and b'. The circuit connection is shown in figure 24*b*; a and a' are in parallel, and b and b' are in parallel. It will be assumed that all four conductors are similar in size, material and design. Then the currents in a and a' will be equal to each other unless the currents of b and b' induce unbalanced voltages in a and a'. The symmetry of the arrangement indicates that this unbalance will not be produced. The fundamental formula may therefore be correctly applied to determine the inductance of phase A (consisting of the two conductors a and a') and of phase B, as follows:

$$L_A = 2 \times 10^{-7} \ln \frac{D_{AB}}{D_{AA}} \text{ h per m;} \tag{62}$$

$$L_B = 2 \times 10^{-7} \ln \frac{D_{AB}}{D_{BB}} \text{ h per m.} \tag{62a}$$

The value of D_{AB}, which is the geometric mean distance from A to B, is the fourth root of the product of the lengths of the four lines joining the centers of a and a' with the centers of b and b'. Since these four lines comprise two equal pairs, D_{AB} may be written:

$$D_{AB} = \sqrt{D\sqrt{D^2 + d^2}}.$$

The value of D_{AA} or D_{BB} is the square root of the product of the distance from a to a' multiplied by the self gmd of a single conductor.

$$D_s = D_{AA} = D_{BB} = \sqrt{dd_s},$$

in which d_s represents the self gmd of one conductor, and D_s is the self gmd of an entire phase, comprising two conductors.

Example. Find the 60-cps reactance per mile of a double-circuit single-phase line arranged as shown in figure 24. Each of the four conductors is a 300,000-cir-mil A.C.S.R., 30 + 7 strands, and the spacings D and d are respectively 10 ft and 5 ft.

Solution.

$$D_m = \sqrt{10\sqrt{125}} = \sqrt{111.8} = 10.57 \text{ ft} = 126.8 \text{ in.}$$

$d_s = 0.290$ in., from table V.

$$D_s = \sqrt{60 \times 0.290} = 4.16 \text{ in.}$$

The reactance may be found from table VII for the ratio $D_m/D_s = 30.4$, or by formula.

$$x_{60} = 0.2794 \log_{10} \frac{126.8}{4.16} = 0.2794 \times 1.483 = 0.415 \text{ ohm per mile.}$$

Another type of double-circuit single-phase arrangement is that shown in figure 25a. If there were no transposition, there would be an inequality of currents in the two individual conductors of each phase, owing to the fact that the currents of the other phase would cause more linkages in the inner conductor of the phase than in the outer conductor, and so set up a circulating component of current around the loop. However, if the sectional configuration of figure 25a is to be used, the conductor currents can be and should be equalized by appropriate transposition. The simplest way of accomplishing this is shown in figure 25b. If the currents were allowed to be unequal in the two conductors, one undesirable result,

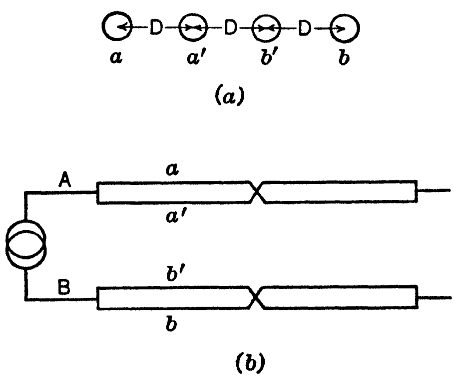


FIG. 25. Double-circuit single-phase line requiring transposition to equalize the currents.

already mentioned, would be an unnecessary increase in the effective resistance of the double circuit to a value more than half that of a single circuit. The computation of the inductance would become more complicated. It is not the purpose of this section to discuss such unbalanced circuits, a treatment of which may, however, be found in article 27.

With the proper transposition, the current density over the entire cross section of each phase is made uniform (or else, if there is non-uniformity within a single conductor, it is taken into account by using the correct equivalent self gmd d_s as already explained).

$$L_A = 2 \times 10^{-7} \ln \frac{D_{AB}}{D_{AA}} \text{ h per m.}$$

$$D_{AB} = \sqrt[4]{D \cdot 2\,D \cdot 2\,D \cdot 3\,D}$$

$$= D\sqrt[4]{12} = 1.860\,D.$$

$$D_s = D_{AA} = \sqrt{Dd_s}.$$

$$L = 2 \times 10^{-7} \ln \frac{1.860\,D}{\sqrt{Dd_s}} = 10^{-7} \ln \frac{3.46\,D}{d_s} \text{ h per m.}$$

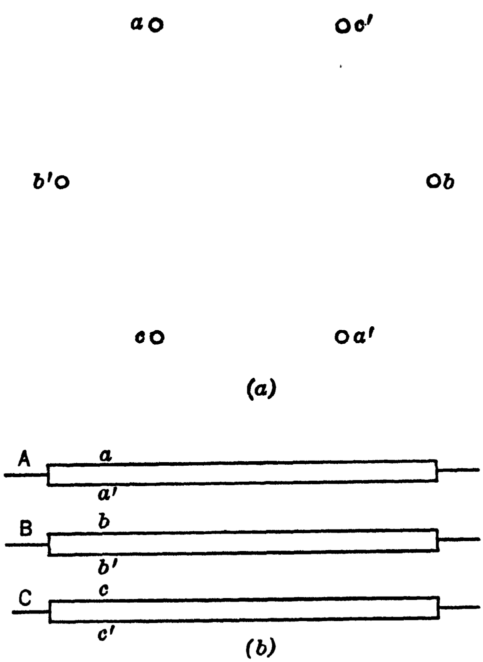


Fig. 26. Double-circuit three-phase line with conductors at corners of a regular hexagon.

Three-phase multicircuit lines. The general formula may be extended to the polyphase multicircuit case. Let us consider first a regular hexagonal arrangement of six equal conductors, indicated in figure 26. Each side of the hexagon is of length D. No transposition is needed to equalize the currents of the two conductors of each phase, because the linkages about each of the two conductors of a phase, produced by the currents of the other two phases, will be equal, owing to the symmetry of the arrangement. The current density over the entire cross section of phase A, consisting of the two individual conductors a and a', will be uniform (or else non-uniformity within a single conductor is taken into account by the use of the correct value of equivalent self gmd d_s); and similarly for the other two phases.

The values of D_m and D_s are computed as follows:

$$D_m = \sqrt[3]{D_{AB}D_{BC}D_{CA}} = \sqrt[12]{D^6(\sqrt{3}\,D)^6} = \sqrt[4]{3}\,D.$$

$$D_s = \sqrt{2\,Dd_s}.$$

$$L = 2 \times 10^{-7} \ln \frac{\sqrt[4]{3}\,D}{\sqrt{2\,Dd_s}} = 10^{-7} \ln \frac{\sqrt{3}\,D}{2\,d_s} \text{ h per m per phase.} \tag{63}$$

$$x_{60} = 0.1397 \log_{10} \frac{\sqrt{3}\,D}{2\,d_s} \text{ ohms per mile per phase.} \tag{63a}$$

The use of this formula may also be justified by comparison with the single-phase case. Considering the linkages about conductor a, it is seen that there are two conductors b' and c' separated from it by the same distance D. The sum of the currents in these two conductors is the negative of that in conductor a (neglecting any zero-sequence component of current in the circuit, which would have to be treated separately, since it would involve a different circuit with neutral return). The combined effect of these two conductors in setting up flux linkages about a will be dependent on the total of the two currents, and not on the individual values of current, since the distance from a is the same for both. The linkages would be the same about a if b' and c' each carried a current of $-\frac{1}{2}\,I_a$. The same reasoning would apply to the effect on linkages about a', and also to the effect of the currents in b and c on the linkages about a and a'. This being so, as far as the inductance of phase A is concerned, the whole circuit may be considered as a single-phase circuit in which the current in one direction is conducted by the two wires a and a', each carrying one-half of the total current; and the current in the other direction by the four wires b, b', c and c', each carrying one-quarter of the total return current. Using this method of finding D_m, we can write:

$$D_m = D_{A < {B \atop C}} = \sqrt[8]{D^4(\sqrt{3}\,D)^4} = \sqrt[4]{3}\,D;$$

the result is the same as obtained before.

Let us consider next the more general hexagonal arrangement shown in figure 27a, in which there is symmetry only about a vertical axis. Transposition is necessary to equalize the inductances (and capacitances) of the different phases, and, depending on the choice of locations for the different phases, it may be necessary to transpose the two wires of each phase with respect to each other in order to equalize their currents. It is always desirable to keep the inductance as low as possible, in order to improve voltage regulation, increase the power limit and improve the power factor. The general inductance formula indicates that inductance decreases with a decrease in D_m and an increase in D_s. It is therefore desirable to *separate the individual conductors of a phase* as far as practicable, and to *keep the distances between phases* as small as practicable. It is indicated then that it is best to

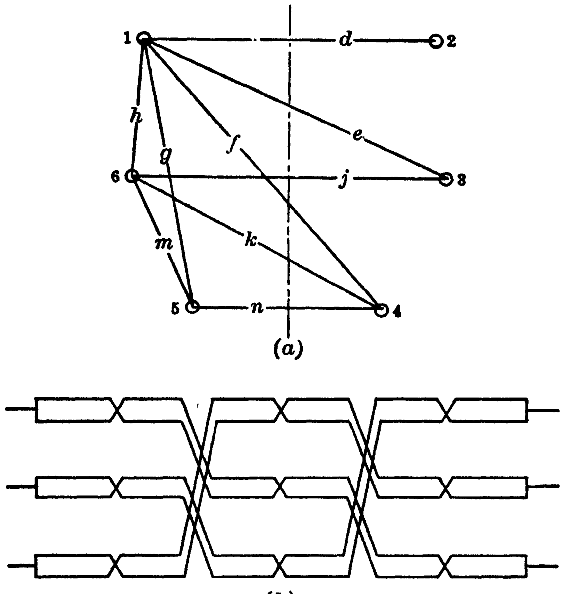


FIG. 27. Double-circuit three-phase line requiring transposition.

put the two conductors of each phase in opposite corners of the hexagon. Phase A should occupy position 1 and 4 for one-third of the distance; positions 2 and 5 for another third, and positions 3 and 6 the final third. One possible transposition cycle is shown in figure 27*b*.

The value of self gmd of phase A will be:

$$D_s = D_{AA} = \sqrt[6]{d_s^3 f^2 j}.$$

The value of mutual gmd from phase A to phases B and C in combination will be:

$$D_m = D_{A < {B \atop C}} = \sqrt[12]{d\, e\, g\, h\, n\, k\, g\, m\, e\, h\, k\, m}$$

$$= \sqrt[12]{d\, e^2\, g^2\, h^2\, k^2\, m^2\, n}.$$

$$x_{60} = 0.1397 \log_{10} \frac{\sqrt[6]{d\, e^2\, g^2\, h^2\, k^2\, m^2\, n}}{d_s \sqrt[3]{f^2 j}} \text{ ohms per mile per phase.}$$

If the hexagon is regular, we have:

$$d = h = m = n = D;$$
$$e = g = k = \sqrt{3}\, D;$$
$$f = j = 2\, D;$$
$$x_{60} = 0.1397 \log_{10} \frac{\sqrt[6]{D^{12}(\sqrt{3})^6}}{d_s \sqrt[3]{D^3}} = 0.1397 \log_{10} \frac{\sqrt{3}\, D}{2\, d_s}$$

ohms per mile per phase.

Instead of transposing the two wires of each phase in the middle of each third of the line shown in figure 27*b*, the two phase conductors could be reversed at the beginning of each new cycle of transposition, but this in effect amounts to doubling the length of the cycle. If the cycle is too long, some of the benefits are lost, on account of the variation in current and voltage from section to section. Common lengths of cycle lie in the range from 12 to 40 miles, and for any given line the desirable length depends largely upon the exposure of parallel communication circuits to inductive interference.

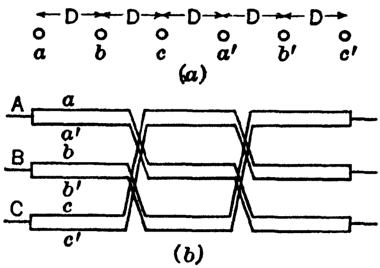


FIG. 28. Double-circuit three-phase line with flat spacing.

There are some double-circuit lines with six conductors all in a plane, as shown in figure 28*a*. The best arrangement of conductors for providing minimum inductance is that shown, in which the two conductors of each phase are separated by two conductors of the other phases. Transposition is necessary to provide balance. Assuming that this is properly taken care of, we can write

$$D_m = D_{A < {B \atop C}} = \sqrt[12]{D \cdot 2\, D \cdot 4\, D \cdot 5\, D \cdot D^2 \cdot (2\, D)^2\, D^2 \cdot 2\, D \cdot 4\, D}$$
$$= D \sqrt[12]{5 \cdot 2^8} = D \sqrt[12]{1280} = 1.815\, D.$$
$$D_s = \sqrt{3\, D d_s}.$$

Example. Find the 60-cps reactance per mile of a double-circuit three-phase line arranged as in figure 28, with transposition. The distance D between centers of adjacent conductors is 20 ft, and each of the six conductors is Anaconda hollow conductor, 650,000 cir-mils, design 560.

Solution. From table VI, $d_s = 0.484$ in.

$$D_m = 1.815\,D = 435.6 \text{ in.}$$

$$D_s = \sqrt{3\,Dd_s} = \sqrt{720 \times 0.484} = 18.68 \text{ in.}$$

$$x_{60} = 0.2794 \log_{10} \frac{435.6}{18.68}$$

$$= 0.2794 \times 1.368 = 0.382 \text{ ohm per mile per phase.}$$

Some lines in use have several circuits. Consider the 18-conductor 6-circuit 3-phase line shown in figure 29. Transposition is required to equalize currents and inductances, and it will be assumed that this is taken care of.

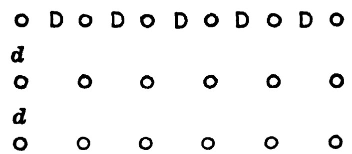


FIG. 29. Six-circuit three-phase line.

First, let the arrangement be that all the conductors at each level comprise one phase. If d_s is the self gmd of one conductor and D_s the self gmd of an entire phase, we have:

$$D_s = \sqrt[18]{d_s^3\, D^{15}\, 1^5\, 2^4\, 3^3\, 4^2\, 5^1}$$

$$= 1.790\,\sqrt[6]{d_s D^5}. \tag{64}$$

The mutual gmd from one phase to the other two phases will be, for the transposed line:

$$D_m = \sqrt[108]{\{d^{12}(D^2+d^2)^{10}\,(4\,D^2+d^2)^8\,(9\,D^2+d^2)^6\,(16\,D^2+d^2)^4 \\ (25\,D^2+d^2)^2\,(2\,d)^6\,(D^2+4\,d^2)^5\,(4\,D^2+4\,d^2)^4\,(9\,D^2+4\,d^2)^3 \\ (16\,D^2+4\,d^2)^2\,(25\,D^2+4\,d^2)^1\}}. \tag{64a}$$

Example. In figure 29, $D = 5$ ft, $d = 5$ ft, and the 18 conductors are all standard concentric stranding 300,000-cir-mil copper cable. Find x_{60} if each level comprises a complete phase.

Solution. From table IV, $d_s = 0.2420$ in.

$$D_s = 1.790\,\sqrt[6]{0.2420 \cdot 60^5} = 42.9 \text{ in.}$$

$$D_m = 60\,\sqrt[108]{1^{12}\,2^{10}\,5^8\,10^6\,17^4\,26^2\,2^6\,5^5\,8^4\,13^3\,20^2\,29^1}$$

$$= 60 \times 2.280 = 136.8.$$

$$x_{60} = 0.2794 \log_{10} \frac{136.8}{42.9}$$

$$= 0.2794 \times 0.504 = 0.1408 \text{ ohm per mile per phase.}$$

Another possible arrangement of the phases, and one which provides a somewhat smaller inductance, is to have the first and fourth vertical columns of conductors comprise one phase; the second and fifth another phase, and the third and sixth columns the remaining phase. With this arrangement, transposition is again necessary, and with this taken into account, we have:

$$D_s = \sqrt[18]{d_s^3\ 4\ d^6\ 27\ D^3 (9\ D^2 + d^2)^2\ (9\ D^2 + 4\ d^2)^1}. \qquad (65)$$

The value of D_s is the same for each transposition section, on the assumption that the phases are interchanged at one-third and two-thirds the way along a transposition section, and therefore the effective value of D_s for the entire line is also the same.

The value of D_m from one phase to the other two will be different in the different transposition sections. The effective value is the gmd of the three component values, and is equal to the following expression:

$$D_m = \sqrt[108]{\{D^{15}\ (2\ D)^{12}\ (4\ D)^6\ (5\ D)^3\ (D^2 + d^2)^{10}\ (4\ D^2 + d^2)^8\ (16\ D^2 + d^2)^4\ (25\ D^2 + d^2)^2\ (D^2 + 4\ d^2)^5\ (4\ D^2 + 4\ d^2)^4\ (16\ D^2 + 4\ d^2)^2\ (25\ D^2 + 4\ d^2)\}}. \qquad (66)$$

Example. With spacings and conductors the same as the previous example, find x_{60} for the condition that the phases are arranged as just described.

Solution.

$$D_s = 60^{5/6} \cdot 0.2420^{1/6}\ \sqrt[18]{4 \times 27 \times 10^2 \times 13}$$
$$= 30.32 \div 1.266 \times 1.932 = 46.3 \text{ in.}$$
$$D_m = 60\ \sqrt[108]{1^{15}\ 2^{12}\ 4^6\ 5^3\ 2^{10}\ 5^8\ 17^4\ 26^2\ 5^5\ 8^4\ 20^2\ 29^1}$$
$$= 60 \times 2.191 = 131.5 \text{ in.}$$
$$x_{60} = 0.2794 \log_{10} \frac{131.5}{46.3}$$
$$= 0.2794 \times 0.4503 = 0.1259 \text{ ohm per mile per phase.}$$

The reactance is more than 10 per cent lower than with the arrangement first considered.

A still better arrangement is to have the conductors of each phase always separated from one another by diagonal distances; thus phase A might comprise the top conductors of the first and fourth columns, the middle conductors of the second and fifth columns, and the bottom conductors of the third and sixth. The computation of this case is left to the student.

27. Unequal Division of Current. If the wires of a multicircuit line are not arranged symmetrically, and also not properly transposed, then the currents will not divide equally between the several conductors of one phase. Unequal division may result also from the use of conductors of different design in parallel with each other.

Consider again the section configuration of figure 25a, but without transposition. Currents in a and a' will differ not only in size but also in phase. We do know from symmetry, however, that the current in a is equal and opposite to the current in b, and that the current in a' is equal and opposite to that in b'. The other condition that enables us to solve for the current in each wire is the fact that the total drop per unit length in a and a' must be the same, since they are connected in parallel.

The total number of linkages about a is

$$0.000\,741\,1\left(I_a \log_{10}\frac{1}{d_s} + I_{a'} \log_{10}\frac{1}{D} + I_{b'} \log_{10}\frac{1}{2\,D} + I_b \log_{10}\frac{1}{3\,D}\right) \text{ linkages per mile,} \quad (67)$$

and since $I_a = -I_b$ and $I_{a'} = -I_{b'}$, the number of linkages about a can be written

$$0.000\,741\,1\left(I_a \log_{10}\frac{3\,D}{d_s} + I_{a'} \log_{10} 2\right) \text{ per mile.} \quad (68)$$

Similarly, the number about a' is found to be

$$0.000\,741\,1\left(I_{a'} \log_{10}\frac{D}{d_s} + I_a \log_{10} 2\right) \text{ linkages per mile.} \quad (69)$$

Let R be the resistance of each conductor in ohms per mile. The condition to be satisfied to make the total impedance drop the same in each of the wires in parallel is that

$$RI_a + j\omega\lambda_a = RI_{a'} + j\omega\lambda_{a'} \text{ volts per mile.} \quad (70)$$

$$RI_a + j\omega\, 0.000\,741\,1\left(I_a \log_{10}\frac{3\,D}{d_s} + I_{a'} \log_{10} 2\right)$$

$$= RI_{a'} + j\omega\, 0.000\,741\,1\left(I_a \log_{10} 2 + I_{a'} \log_{10}\frac{D}{d_s}\right) \text{ volts per mile.} \quad (71)$$

We know also that $I_a + I_{a'} = I$, the total current of the circuit. Solving for the individual currents, we find that

$$I_a = I\left[0.5 - \frac{\log_{10} 3}{\dfrac{4\,R}{j\,0.000\,741\,1\,\omega} + 2\log_{10}\dfrac{3\,D^2}{4\,d_s^2}}\right] \text{ amp;} \quad (72)$$

$$I_{a'} = I\left[0.5 + \frac{\log_{10} 3}{\dfrac{4\,R}{j\,0.000\,741\,1\,\omega} + 2\log_{10}\dfrac{3\,D^2}{4\,d_s^2}}\right] \text{ amp.} \quad (73)$$

It should be noticed that for the direct-current case, where $\omega = 0$, the formulas show that the current divides equally, as the large fraction appearing in each formula becomes zero. When ω has some finite value, the inner conductor carries more current than the outer, this being the simplest case of *skin effect* and *proximity effect.*

Substituting (72) and (73) into one side of (71) gives the total drop per unit length in one of the wires, and hence in both, since they are in parallel.

Example. Find the effective resistance and reactance per mile of an untransposed 60-cps line arranged as in figure 25*a*. The conductors are no. 0000 solid copper having a resistance per mile of 0.264 ohm. The interaxial distance D of adjacent wires is 20 in.

Solution. The drop per mile is found, as indicated, to be

$$I(0.136 + j\,0.358) \text{ vector volts.}$$

The real and imaginary quantities in parenthesis are respectively the effective resistance and reactance in ohms of a mile of line (one conductor of two wires in parallel). The effective resistance is more than half the resistance of one wire on account of the unequal division of current.

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PROBLEMS ON CHAPTER II

Prob. 1–2. Plot a curve showing the variation with spacing of the 60-cps reactance per mile of one wire of a single-circuit single-phase or equilateral three-phase line of no. 0000 solid copper wire. Compute points for spacings of 2, 4, 8, 16 and 32 ft.

Prob. 2–2. Plot a curve showing the variation with conductor area in circular mils, of the 60-cps reactance per mile of one wire of a single-circuit line having 10-ft spacing between conductors. For sizes up to no. 0000 assume that the wire is solid; for larger sizes up to 2,000,000 cir mil it is standard concentric stranded copper.

Prob. 3–2. The 287-kv Boulder Dam-Los Angeles line has hollow interlocking-sector copper conductors of outside diameter 1.40 in. and area of section of 512,000 cir mils. It may be assumed that the shape is a hollow circular cylinder. For most of the length the line runs on single-circuit towers, with 32.5-ft horizontal spacing. Find the reactance per mile at 60 cps. The line is transposed.

Prob. 4–2. A portion of the line from Boulder Dam to Los Angeles is carried on double-circuit towers, on which the six conductors are arranged three high, two conductors at each level. The horizontal spacing is 40.5 ft, and the vertical spacing between adjacent levels is 24.5 ft. Determine the arrangement of the phases which will afford minimum reactance, and show by a sketch the simplest cycle of transposition which will produce balanced phase constants. Calculate the reactance per mile of one phase, at 60 cps.

Prob. 5–2. Derive by integration the inductance in henries per meter of a hollow tubular conductor with parallel return distant D units. Compare your result with the gmd formula for this case. Do they check?

Prob. 6–2. Derive by integration an equation for the inductance of a solid round wire of radius r, whose return conductor circuit is a concentric tube of inside radius r_1 and outside radius r_2. Assume that the relative permeability is unity throughout

Prob. 7–2. Compute the self gmd of a cable consisting of six equal copper strands around a central non-conducting core of the same size as one of the strands. Express in terms of the copper section area in circular measure.

CHAPTER III

SKIN EFFECT

Skin effect is the name given to the tendency of alternating currents to flow with greater density near the outside of conductors. It affects the resistance and to a lesser extent the reactance in large power conductors.

In a long homogeneous wire carrying **continuous** current, the current density is the same in all parts of the cross section. The wire can be supposed to be divided into a number of filaments parallel to the axis, all having equal cross-sectional areas, and hence equal resistances per unit length. In order to have the same ohmic drop per unit length in each filament, the current in each must be the same; in other words, the distribution of current is uniform. Under this condition only is the resistance of a wire correctly given by the usual formula

$$R = \frac{\rho l}{A} \text{ohms},$$

ρ being the resistivity, l the length and A the area of cross section. If two parallel wires are carrying continuous current in opposite directions there is a force of repulsion between them. Since this repulsion is due to the current it might seem that most of the current in one conductor would be repelled to the side farthest away from the other, but this is not the case since the ohmic drop per centimeter length in each filament must be the same.

28. The Cause of Skin Effect. If the current in the wire is **alternating,** there must still be the same drop of voltage per unit length in each filament. Since the drop in this case consists of a voltage of induction in addition to the ohmic drop, and since this voltage of induction is greater at the center of the wire than at the surface because there are more lines of flux surrounding the central part, it follows that the ohmic drop along a filament at the center will be less than along one at the surface. This means a higher current density at the surface, and the unequal current distribution results in a larger power loss for a given rms alternating current than for the same value of direct current flowing in the wire. The **effective alternating-current resistance** (R) of a conductor is defined as the average power loss in the conductor divided by the mean square current, which is the same as the in-phase component of voltage drop divided by the current, for a sine wave. The **skin-effect resistance ratio** (R/R_0) is defined as the quotient of the effective alternating-current resistance divided by the continuous-current resistance. This ratio is very nearly unity for small wires of non-magnetic material at ordinary commercial power frequencies, but increases with the permeability and area of cross section of the conductor, and with the frequency.

29. Alternating-Current Distribution in a Round Wire. Figure 30 represents a cross section of a long wire at a great distance from the return conductor. O is the center of the section, and P is any point in the section. Since the wire possesses circular symmetry, all lines of magnetic flux due to its current are concentric circles with center at O. Let circle PQ represent one of these lines of flux inside the wire. The magnetomotive force acting around this circle is $\int H\,dl$. Since the flux lines are concentric, H is a constant at any fixed radius, and $\int H\,dl$ is equal to $2\,\pi x H$, where x is the radius of the circle PQ. The magnetomotive force acting on any closed circuit is equal to the current passing

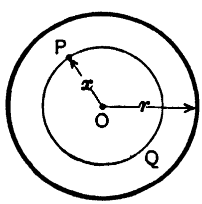

FIG. 30. Cross section of an isolated round wire.

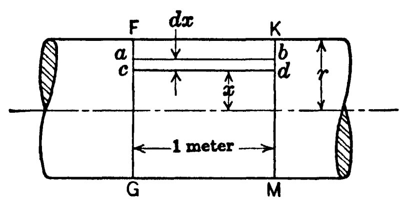

FIG. 31. Isolated round wire.

through or linking the circuit. Let I_x be the total instantaneous current passing through the circle PQ. Then $I_x = 2\,\pi x H$, or

$$H = \frac{I_x}{2\,\pi x} \text{ amp-turns per m.} \tag{74}$$

The flux density at P is independent of the current flowing in the region of the wire outside PQ, as long as the distribution outside has circular symmetry about O.

Consider now a meter length of conductor of radius r, as shown in figure 31. Two assumptions will be made which will cause no appreciable error in any actual line; first, that all the current entering at one end of the bit of wire considered goes out at the other end; in other words, that the charging current of the meter length is negligible in comparison with the conduction current that goes through. Second, it is assumed that the current is everywhere parallel to the axis. If the flow of current is axial there can be no radial component of voltage, and hence FG and KM are equipotential planes. Then the voltage drops along any two lines ab and cd, drawn parallel to the axis, are equal, and the total drop around the rectangle $abdca$ is zero by Kirchhoff's law.

Let

i_x = current density at radius x (amp per sq m);
H_x = magnetizing force at radius x (amp-turns per m);
μ = permeability of wire (assumed uniform and constant);
$B_x = \mu H_x$ = flux density at radius x (webers per sq m);
ρ = resistivity of wire (ohm-meter);
$\omega = 2\pi f$ = angular velocity (radians per sec).

The total current flowing inside the circle of radius x is

$$I_x = \int_0^x 2\pi x i_x \, dx \text{ amperes,} \tag{75}$$

and substituting this value of I_x in (74) there results

$$H_x = \frac{1}{2\pi x}\int_0^x 2\pi x i_x \, dx. \tag{76}$$

Multiplying by x, differentiating and solving for i_x,

$$i_x = \frac{H_x}{x} + \frac{dH_x}{dx}. \tag{77}$$

The flux linking the rectangle $abdca$ is $\mu H_x\,dx$ or $B_x\,dx$, and the emf induced is $\frac{d\phi}{dt} = \mu\,dx\,\frac{dH_x}{dt}$ volts. This must equal the difference in IR drops in ab and cd, which is $\rho\,dx\,\frac{di_x}{dx}$. (Notice that current density multiplied by resistivity is the same as current multiplied by resistance.) Dividing by dx,

$$\rho\frac{di_x}{dx} = \mu\frac{dH_x}{dt}. \tag{78}$$

Differentiating (77) with respect to t,

$$\frac{di_x}{dt} = \frac{1}{x}\frac{dH_x}{dt} + \frac{d^2H_x}{dx\,dt}. \tag{79}$$

Substituting the value of $\frac{dH_x}{dt}$ from (78),

$$\frac{di_x}{dt} = \frac{1}{x}\frac{di_x}{dx}\frac{\rho}{\mu} + \frac{d^2i_x}{dx^2}\frac{\rho}{\mu}.$$

$$\frac{d^2i_x}{dx^2} + \frac{1}{x}\frac{di_x}{dx} = \frac{di_x}{dt}\frac{\mu}{\rho}. \tag{80}$$

This is the general differential equation for any sort of current variation. If the variation is sinusoidal, we write $i_x = i_{mx}\epsilon^{j\omega t}$, i_{mx} being the maximum current density at radius x. $\frac{di_x}{dt}$ now becomes $j\omega i_x$, and equation (80) may be written

$$\frac{d^2 i_x}{dx^2} + \frac{1}{x}\frac{di_x}{dx} - \frac{j\omega\mu}{\rho} i_x = 0. \tag{81}$$

Denoting $\frac{-j\omega\mu}{\rho}$ by n^2, (81) becomes

$$\frac{d^2 i_x}{dx^2} + \frac{1}{x}\frac{di_x}{dx} + n^2 i_x = 0. \tag{82}$$

By differentiating (77) with respect to x and substituting the value of $\frac{di_x}{dx}$ from (78), and making use of the fact that H_x also varies sinusoidally, there is obtained the following equation involving H_x:

$$\frac{d^2 H_x}{dx^2} + \frac{1}{x}\frac{dH_x}{dx} + \left(n^2 - \frac{1}{x^2}\right) H_x = 0. \tag{83}$$

To solve (82) assume that i_x can be expressed in the form of an infinite series of increasing powers of x, as follows:

$$i_x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \tag{84}$$

where the a's are undetermined constants. Performing the required differentiations and substituting in (82), there is obtained

$$\begin{aligned} &2\,a_2 + 3\cdot 2\cdot a_3 x + 4\cdot 3\cdot a_4 x^2 + \cdots \\ &+ \frac{a_1}{x} + 2\,a_2 + 3\,a_3 x + 4\,a_4 x^2 + \cdots \\ &+ n^2 a_0 + n^2 a_1 x + n^2 a_2 x^2 + \cdots = 0, \end{aligned} \tag{85}$$

or

$$a_1\left(\frac{1}{x}\right) + (n^2 a_0 + 4\,a_2) + (n^2 a_1 + 9\,a_3)x + (n^2 a_2 + 16\,a_4)x^2 + \cdots = 0. \tag{86}$$

Each coefficient must be equal to zero. Taking a_0 as the arbitrary constant, it follows that

$$\left.\begin{aligned} a_0 &= a_0, \\ a_1 &= 0, \\ a_2 &= \frac{-a_0 n^2}{2^2}, \\ a_3 &= 0, \\ a_4 &= \frac{-a_2 n^2}{4^2} = \frac{+a_0 n^4}{2^4\,(2!)^2}, \\ a_5 &= 0, \\ &\cdots. \end{aligned}\right\} \tag{87}$$

One solution is therefore

$$i_x = a_0\left(1 - \frac{n^2x^2}{2^2} + \frac{n^4x^4}{2^4\,(2!)^2} - \frac{n^6x^6}{2^6\,(3!)^2} + \frac{n^8x^8}{2^8\,(4!)^2} - \cdots\right). \quad (88)$$

This infinite series is convergent for all finite values of nx, real or imaginary, and is known as the bessel function of the first kind and zero order, abbreviated J_0. The subscript denotes the "order" of the function. Equation (88) could be written thus

$$i_x = a_0 J_0(nx). \quad (89)$$

The J_0 function is one somewhat resembling the cosine function, except that its amplitude decreases continually for increasing values of the argument, if the argument is a real number. Figure 32 is a graph of

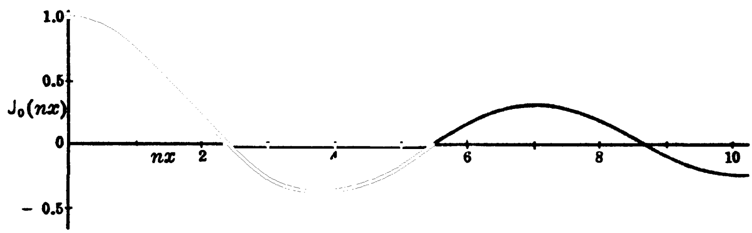

FIG. 32. A plot of the bessel function of the first kind and zero order, $J_0(nx)$, for real values of nx.

$J_0(nx)$ for real values of nx. However, in equation (88) n is not a real but a complex number, being equal to $\sqrt{\dfrac{-j\omega\mu}{\rho}}$, which may be written $jm\sqrt{j}$ where $m = \sqrt{\dfrac{\omega\mu}{\rho}}$. Substituting this value of n in equation (88), there results:

$$i_x = a_0\left(1 + \frac{jm^2x^2}{2^2} - \frac{m^4x^4}{2^4\,(2!)^2} - \frac{jm^6x^6}{2^6\,(3!)^2} + \frac{m^8x^8}{2^8\,(4!)^2} + \cdots\right) \quad (90)$$

It may be observed that alternate terms of the expression in parenthesis in equation (90) are imaginary. Lord Kelvin called the real and imaginary parts of the expression respectively **ber** and **bei** functions, these two words being abbreviations for "**bessel real**" and "**bessel imaginary.**" Thus,

$$i_x = a_0 J_0(nx) = a_0 J_0(jm\sqrt{j}\,x) = a_0(\text{ber}\,mx + j\,\text{bei}\,mx). \quad (91)$$

There is another solution[1] of equation (82), as follows:

$$i_x = b\Big[(\ln 2 - \gamma)\ J_0(jm\sqrt{j}\,x) - \ln jm\sqrt{j}\,x\ J_0(jm\sqrt{j}\,x) + j\frac{m^2x^2}{2^2}$$
$$- \left(1 + \frac{1}{2}\right)\frac{m^4x^4}{2^24^2} - j\left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{m^6x^6}{2^24^26^2} + \cdots\Big]$$
$$= b\,K_0(mx\sqrt{j}) = b(\text{ker}\ mx + j\ \text{kei}\ mx). \tag{92}$$

In the above, γ is the eulerian constant, equal to $0.5772\cdots$. It is the limit of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n$, as n approaches infinity; b is another arbitrary constant. The function K_0 is called a modified bessel function of the second kind and zero order. It is a solution independent of the J_0 solution previously derived, and since the original differential equation (82) is of the second order, it has only two independent solutions. The complete solution for current density is therefore

$$i_x = a_0J_0(nx) + bK_0(-jnx). \tag{93}$$

The arbitrary constants a_0 and b must be determined from the boundary conditions. Since $K_0(0) = \infty$ it follows that the coefficient b must be zero in a solid wire, because the current density cannot be ∞ at the center of the wire, where $x = 0$. Therefore only the first term of (93) remains, and the complete solution, with boundary conditions satisfied, is

$$i_x = i_0J_0(jm\sqrt{j}x), \tag{94}$$

in which i_0 is written in place of a_0, and is equal to the current density at the center of the wire, where $x = 0$. [Note that $J_0(0) = 1$.]

Table VIII presents values of the function representing the current distribution.

TABLE VIII

BESSEL FUNCTIONS OF FIRST KIND AND ZERO ORDER, WITH SEMI-IMAGINARY ARGUMENT

mx	$J_0(jm\sqrt{j}\,x)$ = ber mx + j bei mx	mx	$J_0(jm\sqrt{j}\,x)$ = ber mx + j bei mx
0.0	$1.0000 + j0.0000 = 1.0000 \angle 0.0000°$	3.5	$-1.1936 + j2.2832 = 2.5759 \angle 117.605°$
0.5	$0.9990 + j0.0625 = 1.0010 \angle 3.617°$	4.0	$-2.5634 + j2.2927 = 3.4391 \angle 138.191°$
1.0	$0.9844 + j0.2496 = 1.0155 \angle 14.217°$	4.5	$-4.2991 + j1.6860 = 4.6179 \angle 158.586°$
1.5	$0.9211 + j0.5576 = 1.0768 \angle 31.183°$	5.0	$-6.2301 + j0.1160 = 6.2312 \angle 178.933°$
2.0	$0.7517 + j0.9723 = 1.2286 \angle 52.283°$	5.5	$-7.9736 - j2.7890 = 8.4473 \angle 199.279°$
2.5	$0.4000 + j1.4572 = 1.5111 \angle 74.650°$	6.0	$-8.8583 - j7.3347 = 11.5008 \angle 219.625°$
3.0	$-0.2214 + j1.9376 = 1.9502 \angle 96.518°$		

[1] See Gray and Mathews, *Bessel Functions*, second edition, p. 21, *et seq.*, or Watson, *Theory of Bessel Functions*, p. 57, *et seq.*

If the maximum current density at the center of the wire is taken as unity, and the vector representing this density drawn along the axis of reals, the vector representing the complex current density at any radius x has for its real component the real part and for its imaginary or j component the imaginary part of the function for the corresponding value of mx.

If the effective or rms value of current density at the axis of the wire is taken as unity, the magnitude of the function gives the rms current density at the corresponding radius x. With low frequencies and small non-magnetic wires the function does not depart greatly from the value $1.000\underline{|0.00°}$. For example, in a no. 0000 copper wire carrying 60-cps current ($\mu = 4\pi \times 10^{-7}$, $\omega = 377$, $\rho = 1.724 \times 10^{-8}$ ohm-meter, $r = 0.00584$ m), $mx = x\sqrt{\frac{\omega\mu}{\rho}} = 0.97$ when $x = 0.00584$, the radius. Referring to the table, it is seen that the current density is practically uniform up to this value of mx, or in other words over the whole wire. Suppose that the wire were one of 500,000 circular mils ($r = 0.00898$ m), the frequency being still 60 cps. Then mx at the surface would be 1.49 and the rms current density there would be about 8 per cent greater than at the center.

The table shows also how the currents in different parts of the wire differ in phase. At $mx = 0$ the imaginary part (j bei mx) is zero, and at $mx = 2.85$, ber mx is zero. This means that if the constants μ, ω, ρ and the radius r are such that $mr = 2.85$, the current flowing at the center will be in time quadrature with that at the surface. If $mx = 5.025$, the current at radius x is in direct phase opposition to that flowing at the center. Using the table, the instantaneous current densities, as well as the effective values, can readily be plotted. For example, take a copper wire of radius 1.00 cm carrying a current of frequency 788 cps, and let it be required to plot the instantaneous current densities at intervals of 30 degrees during a half cycle. Since only relative values are desired, the maximum current density at the center can be taken as unity, and the real part of i_x can be taken as the instantaneous current density. Then for the time when $\omega t = 0$ it is necessary only to reproduce the ber mx curve (figure 33a). The origin is at the center of the horizontal line, which represents a diameter. The maximum value of mx is 6.0, from the above data. For the current distribution when $\omega t = \pi/6$, multiply the complex current density at each radius x by $\cos \omega t + j \sin \omega t$, which in this case means multiplying by $\sqrt{3}/2 + j\,(1/2)$, and take the real part for an ordinate of the curve (figure 33b). For $\omega t = \pi/2$, $-$bei mx is the current distribution (figure 33d).

It is interesting to note in these figures how the current seems to start on the outside of the wire and be propagated inward. This is in fact just what does happen. In figure 33*d* the wire is carrying very little net current, but a considerable circulating current is flowing as shown, going one way along the outside and returning along an intermediate path. An idea of how the current is propagated in a smaller wire or at a lower frequency may be obtained by covering equal portions

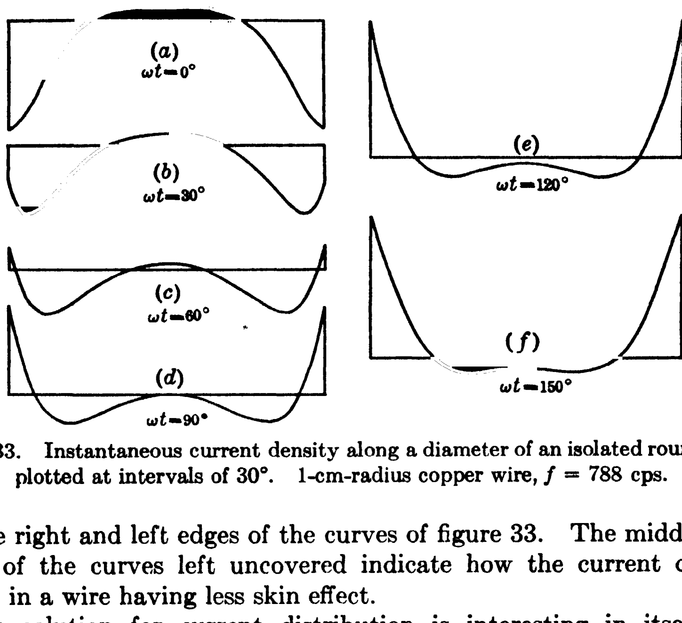

FIG. 33. Instantaneous current density along a diameter of an isolated round wire, plotted at intervals of 30°. 1-cm-radius copper wire, f = 788 cps.

on the right and left edges of the curves of figure 33. The middle portions of the curves left uncovered indicate how the current density varies in a wire having less skin effect.

The solution for current distribution is interesting in itself and necessary as a step in finding the skin-effect resistance ratio, and is of considerable value also in that it indicates which part of the conductor may be removed with least effect on the alternating-current resistance.

30. Skin-Effect Resistance Ratio for Round Wires. The formula for current distribution in a round wire has been derived; it is

$$i_x = i_0 J_0(jm\sqrt{j}\,x). \tag{94}$$

Since it is known that the drop in each filament of the wire must be the same as that in the others, and further that the voltage of self-induction in each filament due to the flux outside the wire is the same, it follows that the sum of the ohmic drop and the induced voltage due to the internal flux must be the same in each filament. Let this sum be denoted by e_i volts per meter. However, as there is no induced voltage due to internal flux on the surface of the wire, we may write simply

$$e_i = i_r\rho = i_0\rho J_0(jm\sqrt{j}\,r) \text{ volts per m,} \tag{95}$$

where i_r denotes the current density at radius r, or at the surface. The total current flowing in the wire is

$$\begin{aligned} I &= \int_0^r 2\,\pi x i_x\,dx \\ &= 2\,\pi i_0 \int_0^r x J_0(jm\sqrt{j}\,x)\,dx \\ &= \frac{2\,\pi i_0 r J_1(jm\sqrt{j}\,r)}{jm\sqrt{j}} \text{ vector amperes,} \end{aligned} \tag{96}$$

where J_1 is a bessel function of the first kind and first order, defined by

$$J_1(z) = \frac{z}{2}\left[1 - \frac{z^2}{2^2 1!\,2!} + \frac{z^4}{2^4 2!\,3!} - \frac{z^6}{2^6 3!\,4!} + \cdots\right]. \tag{97}$$

Formula (96) may be checked by integrating term by term the J_0 series.

The impedance per meter length due to resistance and internal inductance is

$$Z = \frac{e_i}{I} = \frac{jm\sqrt{j}\,\rho J_0(jm\sqrt{j}\,r)}{2\,\pi r J_1(jm\sqrt{j}\,r)} \text{ vector ohms.} \tag{98}$$

If the current density were uniform, the resistance per meter would be

$$R_0 = \frac{\rho}{\pi r^2} \text{ ohms.}$$

The ratio of the internal alternating-current impedance to the direct-current resistance is what we are seeking.

$$\begin{aligned} \frac{Z}{R_0} &= \frac{j\sqrt{j}\,mJ_0(j\sqrt{j}\,mr)\pi r^2}{2\,\pi r J_1(j\sqrt{j}\,mr)} \\ &= \frac{nr}{2}\,\frac{J_0(nr)}{J_1(nr)}. \end{aligned} \tag{99}$$

Letting $Z = R + jL\omega$, where L is the true internal inductance at a frequency $\frac{\omega}{2\,\pi}$, we can write

$$\frac{R}{R_0} = \text{real part of } \frac{nr}{2}\,\frac{J_0(nr)}{J_1(nr)}. \tag{100}$$

In terms of ber and bei functions, (100) becomes

$$\frac{R}{R_0} = \frac{mr}{2}\,\frac{\text{bei}'\,mr\,\text{ber}\,mr - \text{ber}'\,mr\,\text{bei}\,mr}{\text{bei}'^2\,mr + \text{ber}'^2\,mr}, \tag{101}$$

where the primes denote first derivatives with respect to mr. The corresponding formula for the internal inductance ratio is

$$\frac{L}{L_0} = \frac{4}{mr} \frac{\text{ber}' mr \text{ ber } mr + \text{bei}' mr \text{ bei } mr}{\text{ber}'^2 mr + \text{bei}'^2 mr}. \qquad (102)$$

Formulas (101) and (102) were first derived by Lord Kelvin in 1888. Oliver Heaviside had given the solution in a different form a year previously, together with a short table of numerical results. Excellent tables of resistance ratios computed from Lord Kelvin's formula are available in Scientific Paper 169, National Bureau of Standards. The values for skin-effect resistance ratio and inductance ratio given in table IX are taken from that paper.

TABLE IX

SKIN-EFFECT RESISTANCE AND INDUCTANCE RATIOS FOR SOLID ROUND WIRES

mr	$\frac{R}{R_0}$	$\frac{L}{L_0}$	mr	$\frac{R}{R_0}$	$\frac{L}{L_0}$
0.0	1.00000	1.00000	2.5	1.17538	0.91347
0.1	1.00000	1.00000	2.6	1.20056	0.90126
0.2	1.00001	1.00000	2.7	1.22753	0.88825
0.3	1.00004	0.99998	2.8	1.25620	0.87451
0.4	1.00013	0.99993	2.9	1.28644	0.86012
0.5	1.00032	0.99984	3.0	1.31809	0.84517
0.6	1.00067	0.99966	3.5	1.49202	0.76550
0.7	1.00124	0.99937	4.0	1.67787	0.68632
0.8	1.00212	0.99894	4.5	1.86275	0.61563
0.9	1.00340	0.99830	5.0	2.04272	0.55597
1.0	1.00519	0.99741	6.0	2.39359	0.46521
1.1	1.00758	0.99621	7.0	2.74319	0.40021
1.2	1.01071	0.99465	8.0	3.09445	0.35107
1.3	1.01470	0.99266	9.0	3.44638	0.31257
1.4	1.01969	0.99017	10.0	3.79857	0.28162
1.5	1.02582	0.98711	11.0	4.15100	0.25622
1.6	1.03323	0.98342	12.0	4.50358	0.23501
1.7	1.04205	0.97904	13.0	4.85631	0.21703
1.8	1.05240	0.97390	14.0	5.20915	0.20160
1.9	1.06440	0.96795	15.0	5.56208	0.18822
2.0	1.07816	0.96113	20.0	7.32767	0.14128
2.1	1.09375	0.95343	25.0	9.09412	0.11307
2.2	1.11126	0.94482	30.0	10.86101	0.09424
2.3	1.13069	0.93527	40.0	14.39545	0.07069
2.4	1.15207	0.92482	50.0	17.93032	0.05656
			60.0	21.46541	0.04713
			80.0	28.53593	0.03535
			100.0	35.60366	0.02828
			∞	∞	0.00000

Several approximate formulas for low frequencies and for very high frequencies are available, but since tables have been prepared from the exact formulas the others are now of but little use. Experiments have verified the accuracy of the theoretical formula within the limits of experimental error.

31. Approximate Solutions. Effective Depth of Penetration of Current. Lord Rayleigh has shown that for *large wires* or *high frequencies* (so that the skin-effect resistance ratio is large) the effective alternating-current resistance is the same as the direct-current resistance offered by a hollow conductor of the same external dimensions and of thickness equal to

$$\frac{503.3\sqrt{\rho}}{\sqrt{f\mu/\mu_0}}\ \text{m.} \qquad (103)$$

In (103), ρ is in ohm-meters. For standard annealed copper at 20 C this becomes $\frac{6.62}{\sqrt{f}}$ cm, or $\frac{2.61}{\sqrt{f}}$ inches, and for aluminum, $\frac{8.48}{\sqrt{f}}$ cm or $\frac{3.34}{\sqrt{f}}$ inches. This formula holds very closely for any shape of conductor *as long as the radius of curvature of the cross section is everywhere large in comparison with the equivalent depth of penetration* of the current, and is useful for calculating the skin-effect resistance ratios of conductors whose cross sections are complicated geometric figures, as in steel rails, which ordinarily have only a small equivalent depth of penetration. In a case like this the effective conductance is more nearly proportional to the perimeter than to the area of the cross section.

Formula (103) is derived on the assumption of a conducting plate of infinite width. The solution for current distribution and resistance ratio for this case is carried out in the same way as for the round wire, the assumption being made that the flux is everywhere parallel to the surface. The differential equation leads to a solution involving hyperbolic functions of a semi-imaginary instead of bessel functions of a semi-imaginary as in the solution for the circular cylinder. **It should be definitely understood that (103) is not an exact formula, but applies only for small depths of penetration.**

32. Iron and Steel Wire. In order to obtain the solution for skin-effect resistance ratio in a round wire the permeability of the conductor material was assumed to be constant. If the wire is made of iron or steel the permeability varies with the flux density, but an approximate value of the effective resistance and the resistance ratio can be obtained by assuming an average value of permeability to be constant throughout the wire. An average value of permeability in ordinary unannealed galvanized iron or steel wire at the usual current densities is 500, and for annealed iron wire about 800. An average value of resistivity is 0.122 microhm-meter. ($\rho = 1.22 \times 10^{-7}$ in mks units.) Both permeability and resistivity vary within wide limits, and experimental data should be used whenever possible in determining skin effect in magnetic material. Special alloy steels can be made to have exceedingly low relative permeabilities, very nearly unity.

When iron and steel conductors are carrying alternating currents, their increase in effective resistance is due principally to skin effect, but also to some extent to hysteresis.

Table X indicates the magnitude of the resistance ratio for small iron wires.

TABLE X

SKIN-EFFECT RESISTANCE RATIOS OF GALVANIZED IRON WIRE

Grade E.B.B.

B & S Gauge	25 cps	60 cps	2000 cps
4	1.9	3.0	17.
6	1.6	2.5	15.
8	1.3	2.1	12.
10	1.1	1.7	9.7
12	1.0+	1.4	7.9
14	1.0+	1.1	6.0

In addition to its variation with the size of wire and with the frequency, the skin-effect resistance ratio in magnetic conductors varies with the current flowing in the conductor, on account of the change in the permeability. Table XI contains data obtained experimentally at the National Bureau of Standards.

TABLE XI

EFFECTIVE RESISTANCE IN OHMS PER 1000 FT AT 20 C

No. 4 Galvanized Iron Wire, Grade E.B.B.

Frequency	Rms Current in Amperes					
	0.5	1.5	3.0	5.0	7.5	10.0
Direct Current	1.14	1.14	1.14	1.14	1.14	1.14
50 cps	1.23	1.39	1.76	2.32	2.63	2.68
100 cps	1.38	1.71	2.22	3.04	3.63	3.78
500 cps	2.60	3.32	4.48	6.55	8.75	9.15

33. Steel Rails. The most practical method of calculating the skin-effect resistance ratio of a steel rail is to compute the equivalent depth of penetration of current by formula (103). The formula is not strictly applicable on account of the small radius of curvature at certain portions of the section. It has been shown by tests that the actual increase in effective resistance for rails of ordinary configuration may differ as much as 30 per cent from the increase calculated by Rayleigh's formula, using values of permeability obtained in direct-current tests.

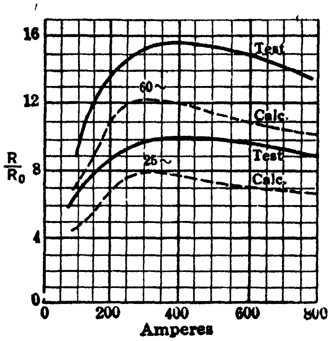

FIG. 34. Variation with current of skin-effect resistance ratio in a typical steel track rail. *Kennelly, Achard and Dana.*

Figure 34 shows the effect of variation of current on the skin-effect resistance ratio in a typical track rail. Contact or "third" rails may be designed to have a smaller ratio by increasing the perimeter for the same area of section, since they are not required to have such high mechanical strength.

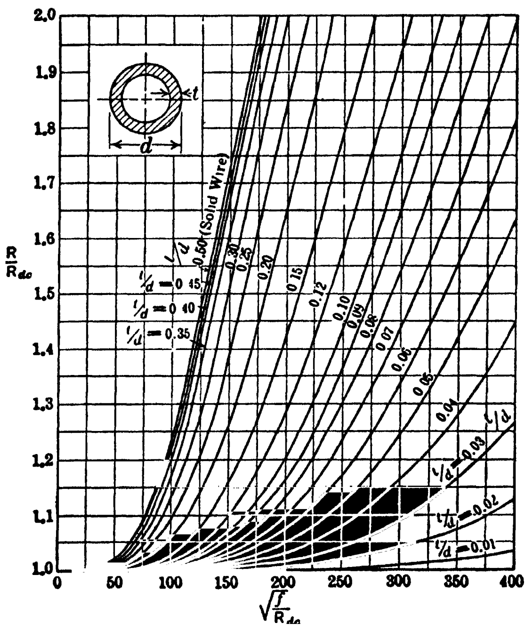

FIG. 35. Skin-effect resistance ratios in tubular conductors. R_{dc} = ohms per 1000 ft. *H. B. Dwight.*

34. Tubular and Strap Conductors. Hollow conductors are being used increasingly for transmission-line and bus conductors. They have the advantage of small skin-effect resistance ratio, diminished inductance and a decreased dielectric gradient as compared to solid conductors of the same area of metal. This last effect may be most important on very high-voltage lines, since it affects the amount of corona loss. The internal impedance of a hollow tube of inside radius q and outside radius r is, per meter length,

$$Z_{t} = \frac{n\rho}{2 \times 10^3 \pi r} \frac{J_0(nr) - \dfrac{J_1(nq)}{G_1(nq)} G_0(nr)}{J_1(nr) - \dfrac{J_1(nq)}{G_1(nq)} G_1(nr)} \text{ ohms.} \qquad (104)$$

In this formula, $n = \sqrt{\dfrac{-j\omega\mu}{\rho}}$. Numerical values of skin-effect resistance ratio for tubes may be obtained more easily from the curves reproduced in figure 35. The values given in the curves and obtained from the formula are in good agreement with experimental results.

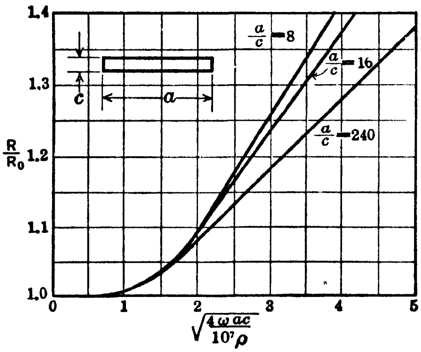

FIG. 36. Skin-effect resistance ratios in strap conductors. Units are meters and ohms. *H. B. Dwight*

Figure 36 gives curves showing the resistance ratio for rectangular conductors. Parts of these curves were computed and parts plotted from experimental results. In computing the value of the abscissas in figure 36, ρ is to be expressed in ohm-meters and is equal to 1.724×10^{-8} for standard annealed copper and 2.826×10^{-8} for aluminum of 61 per cent conductivity. The dimensions a and c are expressed in meters. The same units apply in equation (104).

35. Copper-Clad Steel Wire. The steel core of a copper-clad wire ordinarily carries but little current, largely on account of its high inductance. It is impossible to obtain an exact formula on account of the variation in the permeability. A formula has been worked out on the assumption of constant permeability, but it is very long and complicated, and the results have not been put in engineering form for the large sizes of wire ordinarily used in power transmission. Calculations were made for wire of 30 per cent, 40 per cent and 50 per cent conductivity of sizes up to no. 0 B & S gauge. The calculations check reasonably well with test data. A portion of the data is given in table XII. The resistivity of the steel in the wires tested was 0.126 microhm-meter.

For larger currents the ratio is very slightly increased owing to increased permeability. Very large currents would cause the ratio to decrease again because of saturation.

Although the table does not give resistance ratios for large conductors, they can be calculated from the data given with very little trouble from the relation that the skin-effect resistance ratio in a wire of area A at f cycles per second is the same as for a similar wire of area nA at f/n cycles per second. To illustrate, from the tables the skin-effect resistance ratio for a 30-per cent conductivity no. 6 wire at 3000 cps is seen to be 1.43. No. 0 wire has about four times the area of no. 6, and on this basis we should expect 1.43 to be the resistance ratio of no. 0 at a frequency of 3000/4 or 750 cycles per second. This checks with the value found by interpolation between 1.40 and 1.46, the resistance ratios for 500 cps and 1000 cps, respectively.

TABLE XII

SKIN-EFFECT RESISTANCE RATIOS FOR COPPER-CLAD STEEL WIRES, FOR SMALL CURRENTS

B & S Gauge	Per Cent Conductivity	50 cps	100 cps	500 cps	1000 cps	3000 cps
	30	1.07	1.18	1 40	1.46	
0	40	1.04	1.10	1.22	1.26	
	50	1.02	1.06	1.14	1.16	
	30	1.03	1.10	1.35	1.42	
2	40	1.02	1.05	1.20	1.24	
	50	1 01	1.03	1.12	1.14	
	30	1 00+	1.05	1.29	1.38	1.48
4	40	1.00+	1.03	1.17	1.21	1.26
	50	1.00+	1.01	1.10	1.13	1.16
	30		1.02	1.21	1.32	1.43
6	40		1.01	1.12	1.18	1.24
	50		1 005	1 07	1.11	1.15

36. Stranded Conductors. For stranded conductors in which the separate strands lie parallel to the axis (without spiraling) the skin-effect resistance ratio has been shown by tests to be the same, within the limits of experimental error, as for solid wires having the same cross-sectional area of metal (not the same outside diameter). If the strands are wound spirally the same condition holds very closely at low frequencies such as are used in power work, but for frequencies above 1200 cps on the cables tested the skin-effect resistance ratio was found to increase more rapidly than for solid wires of the same area. If the stranded conductor has a hollow core or a steel core, an approximate value for the skin-effect ratio can be obtained by using the hollow-tube formula or curves and the bimetallic-wire formula or tables respectively.

Test data should be used whenever available.

37. Closely Spaced Wires. The Proximity Effect. The calculation of skin effect in two closely spaced parallel wires is considerably complicated by the fact that the current in one conductor induces eddy currents in the other, and the current distribution is no longer symmetrical about the axis of the wire. This influence of one wire on another is called the *proximity effect.* Some idea of the magnitude of the proximity effect for two parallel solid round wires can be obtained from table XIII. The frequency at which this ratio is 1.100 can be calculated for any size wire from the relation between size and frequency already mentioned.

TABLE XIII

FREQUENCIES AT WHICH RATIO OF EFFECTIVE RESISTANCE TO D-C RESISTANCE IS 1.100, FOR TWO EQUAL COPPER WIRES OF RADIUS 1 CM EACH, FOR VARIOUS INTERAXIAL SPACINGS

Spacing	Frequency	Spacing	Frequency
∞	99.7 cps	4	77.5 cps
10	95.4	3	66.4
6	88.5	2	48.7

The proximity-effect formulas have been checked by tests made by Kennelly, Laws and Pierce at the Massachusetts Institute of Technology and under H. L. Curtis at the National Bureau of Standards, within the limits of experimental error. (See references at end of chapter.)

When the two parallel conductors are stranded, the increase in effective resistance ratio due to the proximity effect is less than in the case of solid wires, and depends on the degree of spiraling and the contact resistance between strands. For such conductors experimental values of resistance should be obtained from the manufacturers.

Solutions have been worked out for the proximity effect in three-phase lines and for parallel tubes. Skin effect in some relatively simple types of stranded conductors has also been solved.

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PROBLEMS ON CHAPTER III

Prob. 1–3. Plot the relative rms current density in a round copper rod of diameter 1 in. carrying 60-cps current. Take the value of current density on the outside as unity.

Prob. 2–3. Compute the value of ber 2 + j bei 2 from the series.

Prob. 3–3. Give the derivation of the formula for skin-effect resistance ratio in round wires in terms of ber and bei functions starting with the formula where it is given in terms of the functions J_0 and J_1.

Prob. 4–3. Give the derivation of the formula for internal inductance ratio in round wires in terms of ber and bei functions, starting with the formula in terms of J_0 and J_1 functions.

Prob. 5–3. What is the skin-effect resistance ratio of a no. 0000 copper wire at 60 cps? What is its internal inductance ratio? By what percentage is the total inductance changed from its d-c value if the spacing of the wires is 3 ft between centers, three-phase equilateral?

Prob. 6–3. Work problem 5–3 for a 500,000-cir-mil copper conductor. (Such a wire would be stranded, but experiment has shown that the solution for a solid wire of equal area of section applies very closely to a stranded concentric lay cable at low frequencies.)

Prob. 7–3. Work problem 6–3 for a 1,000,000-cir-mil conductor of copper.

Prob. 8–3. What is the temperature coefficient of *effective* resistance at 20 C, of a 1,000,000-cir-mil round copper conductor at a frequency of 60 cps?

Prob. 9–3. Work problem 6–3 for a 500,000-cir-mil aluminum conductor.

Prob. 10–3. Plot a curve between skin-effect resistance ratio R/R_0 as ordinates, and the product of circular mils times frequency as abscissas, for round wire of standard annealed copper.

Prob. 11–3. Work problem 10–3 for aluminum of 61 per cent conductivity.

Prob. 12–3. If costs are such that economic considerations indicate that copper and aluminum are equally advantageous for transmitting direct current, which material is preferable for transmitting alternating current on the basis of change in resistance due to skin effect?

Prob. 13–3. Calculate the skin-effect resistance ratio for a no. 10 copper wire at 1,000,000 cycles per second (a) by the exact round-wire formula (or from tables) and (b) by the formula for the equivalent depth of penetration. What is the percentage error?

Prob. 14–3. Work problem 13–3 for a 500,000-cir-mil copper wire at 60 cps.

Prob. 15–3. What is the skin-effect resistance ratio of a rectangular copper bus bar $\frac{1}{2}$ by 4 in. carrying 25-cps current?

Prob. 16–3. What is the skin-effect resistance ratio of an aluminum tube $\frac{1}{16}$ in. thick and of outside diameter 1 in. carrying 50,000-cps current?

Prob. 17–3. What is the effective resistance in ohms per mile at 60 cps of no. 0000 copper-clad steel wire of 45 per cent conductivity?

Prob. 18–3. What is the effective resistance in milliohms per foot of an aluminum bus bar $\frac{1}{8}$ by 5 in. at 25 cps?

Prob. 19–3. A 100-lb (per yard) steel rail has a perimeter of section equal to 24.4 in. Its resistivity is 0.221 microhm-meter and its value of μ/μ_0 may be taken as 480. Calculate the skin-effect resistance ratio at 25 cps.

Prob. 20–3. Work problem 19–3 for a frequency of 60 cps.

Prob. 21–3. A plate of thickness 2 a is infinite in extent in all other directions. Derive the formula for the current density distribution in terms of distance x from the center. (Note: the flux may be assumed to run parallel to the surface of the plate.)

Prob. 22–3. From the current distribution found in problem 21–3, derive the formula for equivalent depth of penetration for high frequencies, equal to

$$\frac{503.3\sqrt{\rho}}{\sqrt{f\mu/\mu_0}} \text{ meters.}$$

Prob. 23–3. Derive equation (88) by the following method of successive approximations. First assume uniform current distribution and the flux distribution produced thereby. Solve for the current-density distribution produced by this flux distribution. Solve again for a new flux distribution, and so on.

CHAPTER IV

THE DIELECTRIC CIRCUIT. CAPACITANCE

Capacitance has little effect upon the normal operation of short, low-voltage lines and feeders, but it does affect very considerably the efficiency, regulation, power factor and voltage distribution on long, high-voltage lines. An understanding of the dielectric circuit around conductors is indispensable to the intelligent study of the phenomena of corona, inductive interference and insulation design. Transient conditions on lines are affected largely by the capacitance. The calculation of capacitance depends upon the following fundamental definitions in the mks system.

38. Units and Definitions. The **coulomb** is the **unit of charge** or quantity of electricity, and is equal to 1 ampere-second. A coulomb of charge, if concentrated at a point, will repel an equal and like point charge 1 meter distant in air, with a force of 9×10^9 joules per meter. The symbol for charge is Q.

Unit field intensity or potential gradient is 1 **volt per meter.**

Unit electric flux is also the **coulomb,** which is the quantity of flux emerging from a coulomb of charge. The symbol for electric flux is ψ.

Electric flux density D is measured in coulombs per square meter. A unit charge immersed in unit electric flux density in air experiences a force of $36\,\pi \times 10^9$ joules per meter.

Space permittivity k_0 is the ratio in free space between electric flux density and potential gradient, and is equal to $10^7/4\,\pi c^2$ or 8.854×10^{-12}. The symbol c represents the velocity of light, and is equal very nearly to 3×10^8 meters per second.

Specific inductive capacity k/k_0 of a substance is the ratio of the dielectric coefficient k of that substance to the value k_0 for free space.

Unit capacitance is the **farad,** which is defined as the capacitance between two conducting bodies which will have a difference of potential of 1 volt if one of the bodies carries a charge of plus 1 coulomb and the other a charge of minus 1 coulomb, all the other conductors in the field being at zero potential. The symbol for capacitance is C.

When there are several charged bodies $a, b, c, \cdots n$, in a field, it is convenient to use capacitance coefficients. A **capacitance coefficient,** such as C_{aj}, is the ratio of the charge on a to the component potential produced electrostatically thereby on j. The **self capacitance** C_{aa} is the ratio of the charge on a body a to its own potential, the potential of all other bodies being zero.

It is convenient also in transmission work to use another and rather special definition of capacitance, that is, **capacitance to neutral,** which is defined as the quotient of the vector a-c charge on one phase divided by the vector voltage of that phase to neutral, the line voltages being balanced.

39. Calculation of Capacitance. A complete, direct solution for the distribution of charge over, and the field and flux distribution between, two conducting bodies separated by an insulating medium is not in general an easy problem, but fortunately it need be attempted only very rarely in transmission work. The charge on the negative conductor must arrange itself so that each infinitesimal element (or, more strictly speaking, each electron) of it is in equilibrium. The same holds true for the charge on the positive conductor. All the charge resides on the surface, which must be an equipotential surface, unless there is a flow of current causing a voltage drop along the conductor. The capacitances which must be calculated in the study of power transmission circuits can for the most part be obtained with sufficient accuracy by considering the charges on conductors to be uniformly distributed over the surfaces. This sort of distribution on a round wire creates the same component field outside the wire as if the charge were concentrated along an infinitesimal filament coincident with the axis of the wire. Since usually the ratio of spacing to conductor radius is of the order of several hundred, it is obvious that even the extreme case of the concentration of all the charge along an element of the surface could affect the component field intensity at a neighboring conductor only a fraction of 1 per cent, as compared with the assumed conditions.

40. Electric Field about a Long Straight Round Wire. If a long, straight, isolated round wire has an electric charge, the field around the wire will have circular symmetry, and the force on a unit charge brought into the field will be in a radial direction. The difference of potential in volts between any two points in the field is equal to the amount of work done in moving a unit charge, 1 coulomb, from one point to the other. The force at any point acting on the unit charge is equal to the potential gradient there in volts per meter, the force being expressed in joules per meter, a unit equal to about 102 grams. The dielectric flux density at a point x meters from the axis of the charged wire (q coulombs per meter) will be

$$D = \frac{q}{2\pi x} \text{ coulombs per sq m.} \tag{105}$$

This is true because we have a total flux of q coulombs per meter length of wire, passing radially through the curved surface of a circular cylinder 1 meter long and having radius x meters. The area of the surface is $2\pi x$.

The force per unit charge in this field, at radius x, which is the same as the field intensity or voltage gradient, is

$$\frac{dE}{dx} = 36\,\pi \times 10^9 D$$

$$= 18 \times 10^9\, q/x \text{ volts per m.} \qquad (106)$$

The difference of potential E_{ab} between any two points p_a and p_b, distant respectively a and b meters from the charged wire, is equal to the integral of the gradient between these two limits. It makes no difference whether or not the two points lie on the same radial line.

$$V_{ab} = 18 \times 10^9 \int_a^b \frac{q\,dx}{x}$$

$$= 18 \times 10^9\, q \ln \frac{b}{a} \text{ volts.} \qquad (107)$$

Equation (107) is a very important and basic one in connection with the determination of charges and capacitances of systems of parallel wires, because by its repeated use in superposition a complete system of equations can be set up.

41. System of *n* Wires. Charges Totaling Zero. Consider the same set of parallel conductors $a, b, c, d, \cdots n$, illustrated in Fig. 7, page 18. Let the respective charges in coulombs per meter be q_a, q_b, q_c, q_d, $\cdots q_n$. Let the radius of each wire be r meters. The difference of potential between any two wires, such as a and j, will be the sum of component potentials of the form of (107), one component being due to each of the n charged wires. Thus:

$$E_{aj} = 18 \times 10^9 \left(q_a \ln \frac{D_{aj}}{r} + q_b \ln \frac{D_{bj}}{D_{ba}} + q_c \ln \frac{D_{cj}}{D_{ca}} + \cdots + q_n \ln \frac{D_{nj}}{D_{na}} \right) \text{ volts.} \qquad (108)$$

If all the various charges were known, this would suffice to determine the voltage E_{aj}; but in a practical problem, the data are not usually known in this way. A line is designed to operate at a specified voltage or set of voltages, and we must normally determine the charges from this known set of voltages. Hence we must write a set of equations of the form of (108) and solve simultaneously for the several charges.

Let it be assumed for the present that the sum of all the charges is equal to zero; that is, there is present no zero-sequence component of charge. This is true or nearly true on any ordinary power line under normal operating conditions. We may write a set of equations such as the following, replacing r_a by D_{aa} and so on:

$$\left.\begin{aligned}
E_{ab} &= 18 \times 10^9 \sum_{j=a}^{n} q_j \ln \frac{D_{jb}}{D_{ja}} \text{ volts;} \\
E_{ac} &= 18 \times 10^9 \sum_{j=a}^{n} q_j \ln \frac{D_{jc}}{D_{ja}} \text{ volts;} \\
E_{ad} &= 18 \times 10^9 \sum_{j=a}^{n} q_j \ln \frac{D_{jd}}{D_{ja}} \text{ volts;} \\
&\vdots \\
E_{an} &= 18 \times 10^9 \sum_{j=a}^{n} q_j \ln \frac{D_{jn}}{D_{ja}} \text{ volts;}
\end{aligned}\right\} \qquad (109)$$

$$\sum_{j=a}^{n} q_j = 0. \qquad (110)$$

There can be written only $n - 1$ independent equations of the form of (109); hence the need of the additional equation (110). There is a wide range of choice in selecting the $n - 1$ equations (109). The relations that exist may be likened to those that exist among the lengths and bearings of lines joining n fixed points in surveying.

42. Two-Wire Line. Figure 37 represents two equal parallel wires which may be designated as a and b, and if their charges are equal we have the two following equations from (109) and (110):

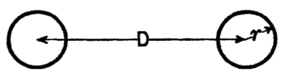

FIG. 37. Cross section of a return circuit of two parallel round wires.

$$E_{ab} = 18 \times 10^9 \left(q_a \ln \frac{D}{r} + q_b \ln \frac{r}{D} \right) \text{ volts;}$$

$$q_a + q_b = 0.$$

Solving for q_a, we find

$$q_a = \frac{E_{ab}}{36 \times 10^9 \ln \dfrac{D}{r}} \text{ coulombs per m.} \qquad (111)$$

The capacitance is the ratio of charge to voltage, and per meter of length it is

$$C_{ab} = \frac{1}{36 \times 10^9 \ln \dfrac{D}{r}} \text{ farads per m} \qquad (112)$$

between the two wires. This may be written

$$C_{ab} = \frac{0.01941}{\log_{10} \frac{D}{r}} \mu\text{f per mile.} \tag{113}$$

The capacitance from one wire " to neutral " in this case is twice the value of the expression of (112) or (113), because it is obtained by dividing the same charge by half as much voltage.

43. Three-Wire Three-Phase Line with Equilateral Spacing. Referring to figure 38, and assuming a balanced set of voltages in accord-

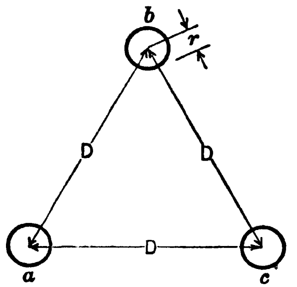

FIG. 38. Cross section of three-phase line with equilateral spacing.

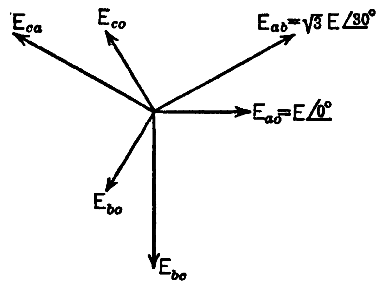

FIG. 39. Balanced three-phase voltage diagram.

ance with the vector diagram of figure 39, we can write the following set of equations, following (109) and (110).

$$E_{ab} = \sqrt{3}\, E \underline{|30^\circ} = 18 \times 10^9 \left(q_a \ln \frac{D}{r} + q_b \ln \frac{r}{D} + q_c \ln 1 \right);$$

$$E_{ac} = \sqrt{3}\, E \underline{|-30^\circ} = 18 \times 10^9 \left(q_a \ln \frac{D}{r} + q_b \ln 1 + q_c \ln \frac{r}{D} \right);$$

$$q_a + q_b + q_c = 0.$$

Solving for q_a, we find

$$q_a = \frac{E}{18 \times 10^9 \ln \frac{D}{r}} \text{ coulombs per m.} \tag{114}$$

The capacitance to neutral is q_a/E, and is equal to

$$C = \frac{1}{18 \times 10^9 \ln \frac{D}{r}} \text{ farads per m.} \tag{115}$$

This is equivalent to

$$C = \frac{0.03883}{\log_{10} \frac{D}{r}} \mu\text{f per mile, to neutral.} \tag{116}$$

At 60 cps, the capacitive susceptance b_{60}, per mile, one phase to neutral, is

$$b_{60} = \frac{14.64 \times 10^{-6}}{\log_{10} \frac{D}{r}} \text{ mho per mile.} \tag{117}$$

44. Three-Phase Line with Unsymmetrical Spacing. In a transposed three-phase overhead line having unsymmetrical spacing, the capacitance or susceptance to neutral may be obtained with very good precision by using (116) or (117) with the spacing D replaced by the geometric mean of the three spacings. This is not, however, an exact relation, as it was in the analogous inductance problem.

Consider the particular case of three wires in a horizontal plane, spaced D between adjacent conductors, and each having radius r, as illustrated in figure 40. Transposition as illustrated in figure 23, page 41, is required to provide balance. Consider the phase voltages to be as shown in figure 39.

For the section in which the conductors are arranged *cab*, the middle conductor a has its potential vector along the axis of reference, and its charge also will lie along the axis of reference. This may be seen from the symmetry of the other two phase conductors both in location and in potential relative to a. Therefore q_a will be a real number. The charges on c and b may be designated as

$$q_c = -\tfrac{1}{2} q_a + jq_q; \tag{118}$$

$$q_b = -\tfrac{1}{2} q_a - jq_q. \tag{119}$$

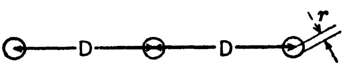

FIG. 40. Cross section of three-phase line in horizontal plane.

The use of the symmetry in this way is not essential, as the general equations may of course be used. It does, however, greatly shorten the work. We can write:

$$\begin{aligned}
E_{ab} &= 18 \times 10^9 \left(q_a \ln \frac{D}{r} + q_b \ln \frac{r}{D} + q_c \ln \frac{2\,D}{D} \right) \\
&= 18 \times 10^9 \left[q_a \ln \frac{D}{r} + (\tfrac{1}{2} q_a + jq_q) \ln \frac{D}{r} + (-\tfrac{1}{2} q_a + jq_q) \ln 2 \right] \\
&= 18 \times 10^9 \left[\frac{3}{2} q_a \ln \frac{D}{\sqrt[3]{2}\, r} + jq_q \ln \frac{2\,D}{r} \right] \\
&= \frac{3}{2} E + j \frac{\sqrt{3}}{2} E.
\end{aligned} \tag{120}$$

Equating the reals to the reals and imaginaries to the imaginaries, and solving,

$$q_a = \frac{E}{18 \times 10^9 \ln \frac{D}{\sqrt[3]{2}\, r}} \text{ coulombs per m,} \tag{121}$$

$$q_q = \frac{\sqrt{3}\,E}{36 \times 10^9 \ln \frac{2\,D}{r}} \text{ coulombs per m.} \tag{122}$$

The values of q_b and q_c in this section may be determined by substituting (121) and (122) into (118) and (119). In the next transposition section, where phase a replaces b, b replaces c and c replaces a, the charge on a will be equal to the charge on b in the first section considered, shifted forward through an angle of 120 degrees. Likewise, in the remaining section q_a will be equal to the value of q_c in the first section shifted backward through an angle of 120 degrees. Designating sections 1, 2 and 3 by the appropriate subscripts, we have:

$$q_{a_1} = \frac{E}{18 \times 10^9 \ln \frac{D}{\sqrt[3]{2}\,r}}.$$

$$q_{a_2} = \left(-\frac{1}{2}\,\frac{E}{18 \times 10^9 \ln \frac{D}{\sqrt[3]{2}\,r}} - j\frac{\sqrt{3}\,E}{36 \times 10^9 \ln \frac{2\,D}{r}}\right)\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{E}{72 \times 10^9 \ln \frac{D}{\sqrt[3]{2}\,r}} + \frac{3\,E}{72 \times 10^9 \ln \frac{2\,D}{r}}$$

$$+ j\left(\frac{\sqrt{3}\,E}{72 \times 10^9 \ln \frac{2\,D}{r}} - \frac{\sqrt{3}\,E}{72 \times 10^9 \ln \frac{D}{\sqrt[3]{2}\,r}}\right).$$

$$q_{a_3} = \left(-\frac{1}{2}\,\frac{E}{18 \times 10^9 \ln \frac{D}{\sqrt[3]{2}\,r}} + j\frac{\sqrt{3}\,E}{36 \times 10^9 \ln \frac{2\,D}{r}}\right)\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{E}{72 \times 10^9 \ln \frac{D}{\sqrt[3]{2}\,r}} + \frac{3\,E}{72 \times 10^9 \ln \frac{2\,D}{r}}$$

$$- j\left(\frac{\sqrt{3}\,E}{72 \times 10^9 \ln \frac{2\,D}{r}} - \frac{\sqrt{3}\,E}{72 \times 10^9 \ln \frac{D}{\sqrt[3]{2}\,r}}\right).$$

The average value of q_a over the whole transposed line is equal to

$$q_{a\,\mathrm{av}} = \frac{q_{a_1} + q_{a_2} + q_{a_3}}{3}$$

$$= \frac{E}{10^9}\left[\frac{1}{12 \ln \frac{D}{\sqrt[3]{2}\,r}} + \frac{1}{12 \ln \frac{2\,D}{r}}\right] \div 3$$

$$= \frac{E}{36 \times 10^9} \left[\frac{\ln \frac{2D}{r} + \ln \frac{D}{\sqrt[3]{2}\, r}}{\ln \frac{D}{\sqrt[3]{2}\, r} \ln \frac{2D}{r}} \right]$$

$$= \frac{E}{18 \times 10^9} \left[\frac{\ln \frac{\sqrt[3]{2}\, D}{r}}{\left(\ln \frac{\sqrt[3]{2}\, D}{r} - \ln 2^{2/3}\right)\left(\ln \frac{\sqrt[3]{2}\, D}{r} + \ln 2^{2/3}\right)} \right]$$

$$= \frac{E}{18 \times 10^9} \frac{1}{\ln \frac{\sqrt[3]{2}\, D}{r} - \frac{\ln^2 2^{2/3}}{\ln \frac{\sqrt[3]{2}\, D}{r}}}$$

$$\doteq \frac{E}{18 \times 10^9 \ln \frac{\sqrt[3]{2}\, D}{r}} \text{ coulombs per m.} \tag{123}$$

$$C \doteq \frac{1}{18 \times 10^9 \ln \frac{\sqrt[3]{2}\, D}{r}} \text{ farads per meter, to neutral.} \tag{124}$$

In the final line of (123), the numerator $\sqrt[3]{2}\, D$ of the fraction under the logarithm sign is equal to the geometric mean of the three spacings. In the next to the last line, the second part of the denominator is small compared to the first part. Table XIV shows the percentage error introduced by ignoring this small term, for various ratios of D/r. The

TABLE XIV

Ratio $\frac{D}{r}$	Per Cent Error
100	0.91
200	0.70
500	0.51
1000	0.42

ratio D/r is commonly equal to several hundred, so the use of the approximate formula results in obtaining a charge or capacitance figure which is too small by approximately 0.5 per cent, for the reason cited. There are, however, other sources of error of comparable size, which will be discussed.

It may be observed that the foregoing derivation applies strictly only to a special unsymmetrical arrangement. The general case with transposition may be worked out in a manner generally similar, but without benefit of the assumptions as to symmetrical charge distribution. In the general case the expressions are so long and complex that it is not considered advisable to present them here. For any particular numerical example the work is relatively simple.

45. Effect of Earth on Capacitance of Transmission Line. Even under balanced line conditions the presence of the conducting earth near a transmission line has a small effect upon the capacitance to neutral. The effect is an increase because the lines of electrostatic flux behave as if the conducting earth had an infinite permittivity, and so more electric flux passes for a given potential difference between conductors than would pass if the conductors were isolated from everything else in free space.

A direct attack on the problem of determining the effect of the earth on capacitance would involve determining the function which represents the variation of charge density along the surface of the earth with distance from the wires, for each case, and then determining the effect of this widely distributed charge on the electric field. Fortunately the **method of images,** suggested by Lord Kelvin, avoids all this complexity.

If the two equal parallel wires of figure 37 have equal and opposite charges, there will be produced a zero-potential plane half way between the two wires, and perpendicular to the plane in which their axes lie. Conversely, then, as Kelvin observed, if we have a configuration comprising a plane and a parallel wire, we may create mathematically the same field if we replace the plane by a hypothetical *image* conductor having a charge equal and opposite to that of the original conductor. If there are several parallel charged wires above a plane, then we may provide a hypothetical image conductor for each, having in each case a charge equal and opposite to the original.

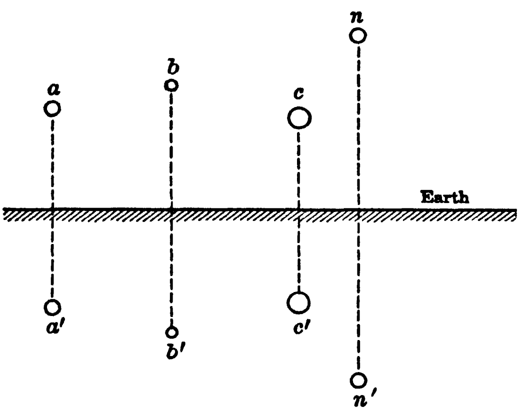

FIG. 41. Arbitrary group of n charged wires, with their images.

Consider the general case illustrated in figure 41. Let us assume that we know the potential difference between each pair of conductors and from each conductor to ground. Then we may write and solve the following equations of type (109) to determine the charges:

$$\left.\begin{aligned}
E_{aa'} &= 2\,E_{ag} = 36 \times 10^9 \sum_{j=a}^{n} q_j \ln \frac{D_{ja'}}{D_{ja}} \text{ volts;}\\
E_{bb'} &= 2\,E_{bg} = 36 \times 10^9 \sum_{j=a}^{n} q_j \ln \frac{D_{jb'}}{D_{jb}} \text{ volts;}\\
E_{cc'} &= 2\,E_{cg} = 36 \times 10^9 \sum_{j=a}^{n} q_j \ln \frac{D_{jc'}}{D_{jc}} \text{ volts;}\\
&\vdots\\
E_{nn'} &= 2\,E_{ng} = 36 \times 10^9 \sum_{j=a}^{n} q_j \ln \frac{D_{jn'}}{D_{jn}} \text{ volts.}
\end{aligned}\right\} \tag{125}$$

Equations (125) comprise a complete set, from which all the charges may be determined.

There may be cases in which the sum of the charges is known to be zero, but where the voltage to ground is not known. Then it is only necessary to write $n - 1$ equations of the form of (109) except that the summation must range over all the images a' to n' as well as the actual conductors a to n. These equations, together with (110), may be solved for all the charges.

The conditions under which the charges per unit length on a group of parallel alternating-current power conductors add to zero are the following:

1. If the entire group is insulated from all other conductors and from ground, and is untransposed; or

2. If the entire group is symmetrically arranged with respect to ground, and has impressed a set of voltages which is balanced and has its neutral point at ground potential. This condition of symmetry is fulfilled only in a cable.

With respect to condition 1, which specifies no transposition, the reason for this restriction is that in a transposed insulated line we do not know that the charge in each transposition section totals zero, even though the fact of the insulation informs us that the charge in all sections must total zero. The electric field equations of the form of (109) and (125) have to be written with respect to one specific cross section of the line at a time. If the line is transposed properly, and balanced voltages impressed, then the neutral will assume ground potential even though it may have no physical connection to ground. The polyphase line capacitances are equivalent to a balanced condenser load, star-connected, on the generator or transformer terminals.

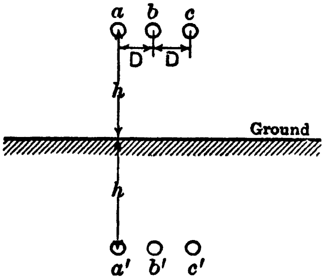

FIG. 42. Three-phase line, with images.

Example. Find the 60-cps susceptance per mile per phase to neutral of a three-phase line arranged as in figure 42. Each conductor is a 715,500-cir-mil steel-reenforced aluminum cable, 54 + 7 strands, with outside diameter 1.036 in. Spacing D is 25 ft, and average height h above ground is 40 ft. The line is transposed.

Approximate solution. Neglecting the effect of ground, we may use (117) if we replace D there by the geometric mean of the three spacings, as pointed out in article 44.

$$b_{60} = \frac{14.64 \times 10^{-6}}{\log_{10} \dfrac{\sqrt[3]{2} \times 300}{0.518}} = \frac{14.64 \times 10^{-6}}{2.863}$$

$$= 5.11 \times 10^{-6} \text{ mho per mile.} \qquad (126)$$

Solution taking into account effect of ground. The following solution cannot be termed exact, but gives a result much more accurate than the preceding. The vector diagram of figure 39 will be followed.

$$\left.\begin{aligned} E_{aa'} &= 2\,E\underline{|0^\circ} &&= 36 \times 10^9 \left[q_a \ln \frac{D_{aa'}}{r} + q_b \ln \frac{D_{ba'}}{D_{ba}} + q_c \ln \frac{D_{ca'}}{D_{ca}} \right]; \\ E_{bb'} &= 2\,E\underline{|-120^\circ} &&= 36 \times 10^9 \left[q_a \ln \frac{D_{ab'}}{D_{ab}} + q_b \ln \frac{D_{bb'}}{r} + q_c \ln \frac{D_{cb'}}{D_{cb}} \right]; \\ E_{cc'} &= 2\,E\underline{|120^\circ} &&= 36 \times 10^9 \left[q_a \ln \frac{D_{ac'}}{D_{ac}} + q_b \ln \frac{D_{bc'}}{D_{bc}} + q_c \ln \frac{D_{cc'}}{r} \right]. \end{aligned}\right\} \qquad (127)$$

Substituting the numerical values of the dimensions, we have:

$$\left.\begin{aligned} 9 \times 10^9\,[15.04\, q_a + 2.421\, q_b + 1.269\, q_c] &= E\underline{|0^\circ}; \\ 9 \times 10^9\,[2.421\, q_a + 15.04\, q_b + 2.421\, q_c] &= E\underline{|-120^\circ}; \\ 9 \times 10^9\,[1.269\, q_a + 2.421\, q_b + 15.04\, q_c] &= E\underline{|120^\circ}. \end{aligned}\right\} \qquad (128)$$

Solving simultaneously (the solution will be indicated by determinants) we find:

$$q_a = \frac{1}{9 \times 10^9} \cdot \frac{\begin{vmatrix} E\underline{|0^\circ} & 2.421 & 1.269 \\ E\underline{|-120^\circ} & 15.04 & 2.421 \\ E\underline{|120^\circ} & 2.421 & 15.04 \end{vmatrix}}{\begin{vmatrix} 15.04 & 2.421 & 1.269 \\ 2.421 & 15.04 & 2.421 \\ 1.269 & 2.421 & 15.04 \end{vmatrix}}$$

$$= \frac{1}{9 \times 10^9} \frac{220.3\,E\underline{|0^\circ} - 33.34\,E\underline{|-120^\circ} - 13.23\,E\underline{|120^\circ}}{3216}$$

coulombs per m.

$$q_b = \frac{1}{9 \times 10^9} \frac{\begin{vmatrix} 15.04 & E\underline{|0^\circ} & 1.269 \\ 2.421 & E\underline{|-120^\circ} & 2.421 \\ 1.269 & E\underline{|120^\circ} & 15.04 \end{vmatrix}}{3216}$$

$$= \frac{1}{9 \times 10^9} \frac{-33.34\,E\underline{|0^\circ} + 224.6\,E\underline{|-120^\circ} - 33.34\,E\underline{|120^\circ}}{3216}$$

coulombs per m.

$$q_c = \frac{1}{9 \times 10^9} \frac{\begin{vmatrix} 15.04 & 2.421 & E\underline{|0^\circ} \\ 2.421 & 15.04 & E\underline{|-120^\circ} \\ 1.269 & 2.421 & E\underline{|120^\circ} \end{vmatrix}}{3216}$$

$$= \frac{1}{9 \times 10^9} \frac{-13.23\,E\underline{|0^\circ} - 33.34\,E\underline{|-120^\circ} + 220.3\,E\underline{|120^\circ}}{3216}$$

coulombs per m.

The average value of q_a in the three transposition sections may be written, following the method used in article 44:

$$q_{a\,\mathrm{av}} = (q_a + q_b\underline{|120^\circ} + q_c\underline{|-120^\circ}) \div 3$$

$$= \frac{E}{9 \times 10^9} \frac{\begin{array}{c}(220.3 + 224.6 + 220.3)\underline{|0^\circ} + (-33.34 - 33.34 - 13.23)\underline{|-120^\circ} \\ + (-13.23 - 33.34 - 33.34)\underline{|120^\circ}\end{array}}{3 \times 3216}$$

$$= \frac{E}{9 \times 10^9} \frac{745.2}{9648} = 8.58 \times 10^{-12}\,E \text{ coulomb per m.} \qquad (129)$$

The indicated capacitance to neutral is therefore

$$C = 8.58 \times 10^{-12} \text{ farad per m,} \qquad (130)$$

which is equivalent, there being 1609 meters to the mile, to

$$C = 1.380 \times 10^{-8} \text{ farad per mile.} \qquad (131)$$

The susceptance at 60 cps is 377 times the capacitance, so we have, finally

$$b_{60} = 5.21 \times 10^{-6} \text{ mho per mile to neutral.} \qquad (132)$$

This is 2.0 per cent larger than the approximate result (126), and the difference represents principally the effect of ground.

It is only when the maximum spacing between conductors is comparable with the height above ground that the effect of the ground on the capacitance is appreciable.

46. Capacitance of Multicircuit Polyphase Lines. The multicircuit line is merely a special case of the n-conductor arrangement already discussed, and is amenable to the same treatment. There may be conditions of symmetry which will make possible short cuts in the work.

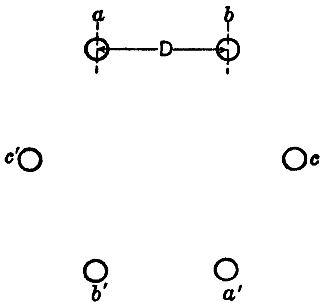

FIG. 43. Double-circuit three-phase line with regular hexagonal arrangement.

Example. The double-circuit three-phase line of figure 43 has the balanced voltages of figure 39 impressed. Find the capacitance per phase to neutral.

From the symmetry which is seen to exist, we may assume that the charges will be balanced, as indicated by the following expressions:

$$q_a = q_{a'};$$
$$q_b = q_{b'} = q_a \underline{|-120°};$$
$$q_c = q_{c'} = q_a \underline{|120°}.$$

There is only one unknown, and so we need write but a single equation. This may be:

$$E_{ab} = \sqrt{3}\, E \underline{|30°} = 18 \times 10^9 \left[q_a \ln \frac{D}{r} + q_a \underline{|-120°} \ln \frac{r}{D} + q_a \underline{|120°} \ln \frac{D}{\sqrt{3}\, D} + q_a \ln \frac{\sqrt{3}\, D}{2\, D} + q_a \underline{|-120°} \ln \frac{2\, D}{\sqrt{3}\, D} + q_a \underline{|120°} \ln \frac{\sqrt{3}\, D}{D} \right]$$

$$= 18 \times 10^9\, q_a \underline{|30°} \sqrt{3} \ln \frac{\sqrt{3}\, D}{2\, r} \text{ volts.} \tag{133}$$

The capacitance for the whole phase of two conductors is $2\, q_a/E$, and is equal to

$$C = \frac{1}{9 \times 10^9 \ln \dfrac{\sqrt{3}\, D}{2\, r}} \text{ farads per m, to neutral.} \tag{134}$$

The capacitance per mile is:

$$C = \frac{0.07765}{\log_{10} \dfrac{\sqrt{3}\, D}{2\, r}} \mu f \text{ to neutral, per mile.} \tag{135}$$

Compare these formulas with (63) and (63a), which give the inductance and reactance for the same arrangement.

$$\begin{cases} L = 10^{-7} \ln \dfrac{\sqrt{3}\,D}{2\,d_s} \text{ henries per m;} & (63) \\[2ex] C = \dfrac{1}{9 \times 10^9 \ln \dfrac{\sqrt{3}\,D}{2\,r}} \text{ farads per m.} & (134) \end{cases}$$

Examining these two expressions in detail, we may note that the inductance depends upon the self gmd of the wire, whereas the capacitance depends upon its radius. This difference is logical because the electric field ends at the surface of the wire, while the magnetic field extends inside and produces there some linkages and internal inductance.

Another difference is that the logarithm term is in the numerator of the inductance expression and in the denominator of the capacitance expression.

The third and final difference is by a factor of 9×10^{16} in the coefficient of the logarithm term. This number is the square of the velocity of light in meters per second. (Actually the value for the velocity of light in vacuum is $[2.99796 \pm 0.00004]\ 10^8$ meters per second, as determined by Michelson in 1926.)

This relation between the capacitance and inductance formulas is not a fortuitous one in the present case, but is a general relation which is capable of being very useful. We may write for the inductance:

$$L = L_e + L_i \text{ henries per meter,} \tag{136}$$

in which L_e represents the inductance due to flux external to the wire and L_i represents the inductance due to internal flux and linkages. Then we may write

$$C = \frac{1}{c^2 L_e} \text{ farads per m,} \tag{137}$$

where c represents the velocity of light, 3×10^8 meters per second.

Another way of stating the reciprocal relation which exists is to compare the following gmd formulas. We have for inductance:

$$L = 2 \times 10^{-7} \ln \frac{D_m}{D_s} \text{ henries per m,} \tag{138}$$

in which D_s represents the self gmd of the current of one phase to itself, which is ordinarily the self gmd of the section area. For the capacitance we may write:

$$C = \frac{1}{c^2 \times 2 \times 10^{-7} \ln \frac{D_m}{D_{ss}}} \text{ farads per m,} \tag{139}$$

$$C = \frac{0.03883}{\log_{10} \frac{D_m}{D_{ss}}} \mu f \text{ per mile, to neutral,} \tag{139a}$$

in which D_{ss} represents the self gmd of the charge of one phase to itself, which is ordinarily the self gmd of the perimeters, since the charge resides on the surface. It is to be noted that the self gmd of a circular line is equal to its radius.

TABLE XV

60–CPS SUSCEPTANCE PER MILE AS A FUNCTION OF D_m/D_{ss}

(For single-circuit line, D_{ss} = Conductor radius)

D_m/D_{ss}	b_{60} in Micro-mhos per Mile to Neutral	D_m/D_{ss}	b_{60}	D_m/D_{ss}	b_{60}	D_m/D_{ss}	b_{60}
2500	4 31	1450	4.63	400	5.62	30	9.91
2450	4.32	1400	4.65	350	5.75	25	10.47
2400	4.33	1350	4.68	300	5.91	20	11.25
2350	4.34	1300	4.70	250	6.11	15	12.45
2300	4.36	1250	4.73	200	6.36	10	14.64
2250	4.37	1200	4.75	190	6.43	9	15.34
2200	4.38	1150	4.78	180	6.49	8	16.21
2150	4.39	1100	4.81	170	6.56	7	17.32
2100	4.41	1050	4.84	160	6.64	6	18.81
2050	4.42	1000	4.88	150	6.73	5	20.94
2000	4.43	950	4.92	140	6.82	4	24.32
1950	4.45	900	4.96	130	6.92		
1900	4.47	850	5 00	120	7.04		
1850	4.48	800	5.04	110	7.17		
1800	4.50	750	5.09	100	7.32		
1750	4.51	700	5.15	90	7.49		
1700	4.53	650	5.20	80	7.70		
1650	4.55	600	5.27	70	7.93		
1600	4.57	550	5.34	60	8.23		
1550	4.59	500	5.42	50	8 62		
1500	4.61	450	5.52	40	9.14		

For a general proof of the relation, which applies strictly to balanced symmetrical lines and approximately to transposed unsymmetrical lines, we need only go back to the basic expressions for the fields about a single wire. We have for the linkages per meter between two lines a and b which run in air parallel to a long straight wire carrying I amperes:

$$\lambda_{ab} = 2 \times 10^{-7} I \ln \frac{D_b}{D_a} \text{ linkages per m.} \qquad (140)$$

If the wire also has a charge of q coulombs per meter, it will produce between a and b a difference of potential equal to

$$E_{ab} = 18 \times 10^9 q \ln \frac{D_b}{D_a} \text{ volts.} \qquad (141)$$

The same relation holds between $\frac{\lambda_{ab}}{I}$ in (140) and $\frac{q}{E_{ab}}$ in (141) which has already been noted. These may be considered as the building blocks from which are constructed the final formulas for inductance and capacitance, and unless there is a different ratio between I and q for different wires, the same relation holds in the comparison of each component, and hence also for the whole.

The effect of ground upon capacitance is not taken into account by the method just described. If the effect of ground is negligible, table XV may ordinarily be used in the determination of numerical values of susceptance per mile at 60 cps, for either single-circuit or multicircuit lines. If the line is single-circuit, D_{ss} is to be taken equal to the radius of the conductor.

47. Conductor Sizes. If the conductor is a solid round wire, it is obvious that the electric field begins at a distance from the center equal to the radius of the wire. If the conductor is stranded, it is not quite exact to use the extreme outer radius, but the error introduced by so doing is very small. The cross section of a seven-strand cable is shown in figure 12, page 25. The electric field set up by a charge on this cable is illustrated in figure 44. It is easily seen that a given quantity of electric flux would cause less potential drop in passing outward from the large dotted circle than it actually causes when it originates as

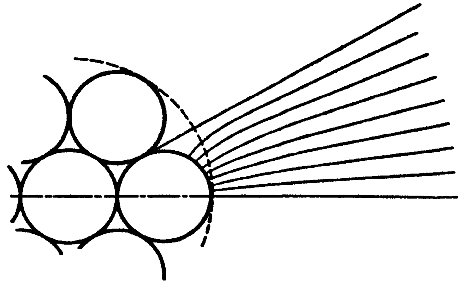

FIG. 44. Electric flux distribution near a seven-strand cable.

shown along the surface of the strands. The difference is small, however, and for a seven-strand cable could be corrected for by using an equivalent radius about 4 per cent smaller than the actual outside radius. This would represent a decrease in capacitance of about 1 per cent with normal overhead spacing. The correction for larger cables having 19 or more strands would be much less. It may be noted that the presence of towers and poles, ground wires, etc., increases the capacitance. If the outside diameter is used in the calculation of capacitance, and these other small effects neglected, there will be at least a partial cancellation of the small errors, and the results will be close enough for almost any practical purpose.

In table XVI are given the outside radii of some of the common designs of transmission conductors.

Example. Find the 60-cps susceptance to neutral per mile of a double-circuit three-phase line arranged as in figure 28, with transposition as shown. The distance D between centers of adjacent conductors is 20 ft, and each of the six conductors is Anaconda hollow conductor, 605,000 cir mil, design 560.

Solution. From table VI, the radius is 0.553 in. (**Caution: do not use d_s in capacitance calculations.**) The value of D_m, from article 26, is

$$\begin{aligned} D_m &= \sqrt[12]{\{D \cdot 2\,D \cdot 4\,D \cdot 5\,D \cdot D^2 \cdot (2\,D)^2\,D^2 \cdot 2\,D \cdot 4\,D\}} \\ &= D\sqrt[12]{1280} = 1.815\,D \\ &= 435.6 \text{ in.} \\ D_{ss} &= \sqrt{3\,Dr} = \sqrt{3 \times 240 \times 0.553} \\ &= 20.0 \text{ in.} \end{aligned}$$

The ratio of D_m/D_{ss} is $435.6/20 = 21.78$, and from table XV the value desired may be read:

$$b_{60} = 10.97 \text{ micromhos per mile, to neutral.}$$

The same result is obtained by using the general formula

$$\begin{aligned} b_{60} &= \frac{14.64}{\log_{10}\dfrac{D_m}{D_{ss}}} = \frac{14.64}{1.337} \\ &= 10.97 \text{ micromhos per mile, to neutral.} \end{aligned}$$

TABLE XVI

Outside Radius of Common Transmission Conductors

Solid or Concentric Stranded

Cir Mils	B & S Gauge	Solid	Stranded			
		Radius (inches)	Standard		Flexible	
			Strands	Radius (inches)	Strands	Radius (inches)
2,000,000			127	0.816	169	0.816
1,900,000			127	0.795	169	0.795
1,800,000			127	0.774	169	0.774
1,700,000			127	0.752	169	0.752
1,600,000			127	0.730	169	0.730
1,500,000			91	0.706	127	0.707
1,400,000			91	0.682	127	0.683
1,300,000			91	0.658	127	0.658
1,200,000			91	0.632	127	0.632
1,100,000			91	0.605	127	0.605
1,000,000			61	0.576	91	0.577
950,000			61	0.562	91	0.562
900,000			61	0.547	91	0.547
850,000			61	0.531	91	0.532
800,000			61	0.516	91	0.516
750,000			61	0.499	91	0.500
700,000			61	0.482	91	0.483
650,000			61	0.465	91	0.465
600,000			61	0.447	91	0.447
550,000			61	0.428	91	0.428
500,000			37	0.407	61	0.408
450,000			37	0.386	61	0.387
400,000			37	0.364	61	0.365
350,000			37	0.341	61	0.341
300,000			37	0.315	61	0.316
250,000			37	0.288	61	0.288
211,600	0000	0.2300	19	0.264	37	0.265
167,800	000	0.2048	19	0.235	37	0.236
133,100	00	0.1824	19	0.209	37	0.210
105,500	0	0.1625	19	0.187	37	0.187
83,700	1	0.1447	19	0.166	37	0.167
66,400	2	0.1288	7	0.146	19	0.148
52,600	3	0.1147	7	0.130	19	0.132
41,700	4	0.1022	7	0.116	19	0.117
33,100	5	0.0910	7	0.103	19	0.105
26,250	6	0.0810	7	0.092	19	0.093
20,820	7	0.0722	7	0.082	19	0.083
16,510	8	0.0643	7	0.073	19	0.074
13,090	9	0.0510				
10,380	10	0.0404				

TABLE XVI — *Continued*

STEEL-REENFORCED ALUMINUM CABLE

Cir Mils Aluminum	Stranding		Radius (inches)
	Aluminum	Steel	
1,590,000	54	7	0.772
1,510,500	54	7	0.753
1,431,000	54	7	0.733
1,351,500	54	7	0.712
1,272,000	54	7	0.691
1,192,500	54	7	0.669
1,113,000	54	7	0.646
1,033,500	54	7	0.623
954,000	54	7	0.598
900,000	54	7	0.581
874,500	54	7	0.573
795,000	54	7	0.547
715,500	54	7	0.518
666,600	54	7	0.500
636,000	54	7	0.489
605,000	54	7	0.477
556,500	30	7	0.477
556,500	26	7	0.456
518,000	42	19	0.500
500,000	30	7	0.452
477,000	30	7	0.442
477,000	26	7	0.423
397,500	30	7	0.403
397,500	26	7	0.386
336,400	30	7	0.371
336,400	26	7	0.355
300,000	30	7	0.350
300,000	26	7	0.335
266,800	26	7	0.316
266,800	6	7	0.317
211,600	6	1	0.282
167,800	6	1	0.251
133,100	6	1	0.224
105,500	6	1	0.199
83,700	6	1	0.178
66,400	6	1	0.158
52,600	6	1	0.141
41,700	6	1	0.125
33,100	6	1	0.112
26,250	6	1	0.099

ANACONDA HOLLOW CONDUCTOR

OUTSIDE DIAMETERS OF ANACONDA HOLLOW CONDUCTORS ARE GIVEN IN TABLE VI.

(References are on page 92.)

PROBLEMS ON CHAPTER IV

Prob. 1–4. Plot a curve showing the variation with spacing of the 60-cps capacitive susceptance per mile to neutral of one wire of a single-circuit single-phase or equilateral three-phase line of no. 0000 solid copper wire. Compute points for spacings of 2, 4, 8, 16 and 32 ft.

Prob. 2–4. Plot a curve showing the variation with conductor area in circular mils, of the 60-cps capacitive susceptance per mile to neutral of one wire of a single-circuit line having 10-ft spacing between conductors. For sizes up to no. 0000 assume that the wire is solid; for larger sizes up to 2,000,000 cir mils, it is standard concentric stranded copper.

Prob. 3–4. The 287-kv Boulder Dam-Los Angeles line has hollow copper conductors 1.40 in. in outside diameter. For most of the length, single-circuit towers are used, and the conductors are suspended in a horizontal plane, 32.5 ft on centers. They are transposed. What is the value of capacitive susceptance per mile to neutral at 60 cps? Neglect the effect of the ground on capacitance.

Prob. 4–4. Compute the 60-cps susceptance per mile of the single-circuit Boulder Dam-Los Angeles line, taking into account the effect of ground. The average height of the conductors above ground is 50 ft. What is the percentage effect of ground on the susceptance?

Prob. 5–4. Compute the 60-cps susceptance of the double-circuit section of the Boulder Dam-Los Angeles line, which has horizontal spacing of 40.5 ft and vertical spacing between adjacent levels of 24.5 ft. The line is transposed, the diagonally opposite corner conductors being of the same phase. The effect of ground is to be neglected.

Prob. 6–4. What is the capacitance per meter between two parallel round wires spaced 5 ft on centers, one having a radius equal to 0.10 in. and the other a radius of 0.20 in.?

Prob. 7–4. Two parallel no. 0000 solid wires are 10 ft apart and 20 ft above ground in a horizontal plane. A potential difference of 50,000 volts at 60 cps exists between the wires. Find the charging current per mile if the neutral is at ground potential; that is, the voltage from either wire to ground has a magnitude of 25,000 volts.

Prob. 8–4. Compute the 60-cps charging current per mile for each of the two conductors of problem 7–4, if one of the wires is at ground potential and the other 50,000 volts above ground.

Prob. 9–4. Compute and plot the equipotential lines in a cross-section view of a circuit of two parallel wires of radius 1 cm each, and interaxial spacing 1 m. The wires have 100 volts difference of potential, one being at +50 and the other −50 volts. Draw curves for each 10 volts difference of potential.

Prob. 10–4. Derive the formula for the capacitance per meter and per mile between the round conductor and concentric sheath of a single-phase lead-sheathed insulated cable. Let r be the radius of the conductor and R the inside radius of the lead sheath. Express in terms of the specific inductive capacity k/k_0 of the insulating material.

Prob. 11-4. A single-circuit three-phase transposed line with 25-ft horizontal spacing of conductors, and average height above ground 40 ft, has conductor diameter 1.036 in. The conductors are transposed. The line operates without a grounded neutral, and one of the conductors becomes accidentally grounded. The line-to-line voltages remain approximately normal. Determine the charging current per mile of each phase in terms of normal line-to-neutral voltage E. The frequency is 60 cps.

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CHAPTER V

STEADY-STATE CURRENTS AND VOLTAGES ON POWER TRANSMISSION LINES

The electrical problems encountered in dealing with transmission lines are those of regulation, efficiency and stability, power-factor correction and voltage control, corona and insulation, and inductive interference with neighboring communication circuits. In addition to these problems dealing with conditions of normal operation are those of providing devices to protect against damage due to unusual conditions, such as lightning disturbances or short circuits.

The problems of normal operation are attacked in different manners depending on the constants (resistance, inductance, capacitance, leakance) of the line. The reason for this is that in solutions dealing with short lines certain assumptions can be made which considerably simplify the problem and which at the same time yield solutions of engineering accuracy. If these same assumptions were made in dealing with long lines, errors of too great magnitude would be introduced. The approximations spoken of are in the methods of taking into account the effect of the capacitance of the lines. The distributed leakance is usually a negligible factor.

The capacitance between two long parallel wires, or from one wire to neutral in the three-phase case, is directly proportional to the length of the wires, other things being equal. The voltage which is used in transmitting power over a line is governed by economic considerations, and varies very roughly as the length of the line. The charging current of the line varies therefore about as the square of the length. The voltage drop due to the charging current varies directly as the length of line through which the current flows, for a given current. It is evident that there is an enormous difference in the relative effect of the capacitance in two lines of, say, 20 and 200 miles respectively in length.

Nevertheless, it is a good plan, and a safe one, always to use a method which accurately takes into account the distributed constants. Then if some less accurate method would have yielded an acceptable result, nothing is lost except a few minutes of time. The important gain is in knowing that results are accurate. There are, however, many short low-voltage lines in which a preliminary study will indicate that the effect of capacitance and charging current is absolutely negligible.

48. Very Short Low-Voltage Lines. When the charging current is neglected the problem becomes simply one of a series circuit. The resistance and reactance of the line are distributed, but this has no effect on the total drop in voltage due to a given current flowing through them. The effect is the same as if the current flowed first through a resistanceless reactance equal to the total reactance of the line, and then through a reactanceless (or non-inductive) resistance equal to the total resistance of the line. In a single-phase line there is the same current in each wire, so the total drop may be calculated at once by multiplying the impedance of **both** wires by the current. The vector diagram is shown in figure 45. The voltage formula is

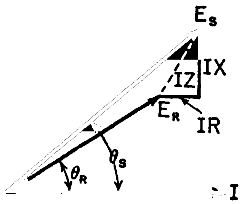

FIG. 45. Vector diagram of a single-phase transmission line, with charging current neglected.

$$E_S = E_R + IZ \text{ vector volts} \tag{142}$$

where Z is the vector impedance of **both** wires.

The numerical or absolute value of the sending-end voltage E_S, in terms of the receiving-end voltage E_R and the drop, is evidently (see figure 45) given by

$$E_S{}^2 = (E_R \cos\theta_R + IR)^2 + (E_R \sin\theta_R - IX)^2 \text{ volts}^2. \tag{143}$$

θ_R is the power-factor angle at the receiving end taken as + for a leading current and − for a lagging; R, X and Z are the constants of **both** conductors.

In a **three-phase** transmission line the drop in voltage between two wires cannot be found by multiplying the current per wire by the impedance of both wires for the reason that the currents in the two wires, although equal numerically if the load is balanced, are not in phase. The simplest way is to treat each wire separately. The current in a wire is displaced in phase from the voltage to neutral of that wire by the power-factor angle. This is analogous to the single-phase line. The only difference lies in the fact that neutral potential is the same at both ends, and so the total drop in voltage to neutral is that which occurs in one wire. The formula (in vector form) is

$$\frac{E_S}{\sqrt{3}} = \frac{F_R}{\sqrt{3}} + IZ \text{ vector volts,} \tag{144}$$

the size of E_S being given by

$$\frac{E_S{}^2}{3} = \left(\frac{E_R}{\sqrt{3}}\cos\theta_R + IR\right)^2 + \left(\frac{E_R}{\sqrt{3}}\sin\theta_R - IX\right)^2 \text{ volts}^2; \tag{145}$$

in these last two equations E_S and F_R are the voltages between lines at the sending end and receiving end respectively. The circuit is shown in figure 46 and the vector diagram for balanced conditions in figure 47.

The foregoing relations are really no more than an application of Ohm's law for the alternating-current circuit.

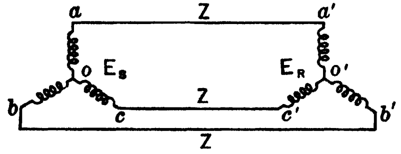

Fig. 46. Three-phase transmission line.

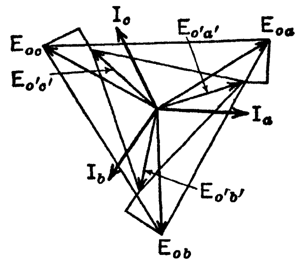

Fig. 47. Vector diagram of a balanced three-phase transmission line, with charging current neglected.

Example. A 10-mile 60-cps single-circuit three-phase line of no. 0000 solid copper conductors equilaterally spaced 5 ft on centers is to deliver 2500 kw at 0.80 power factor, lagging current, to a balanced load at 11,000 volts between lines. What must be the sending-end voltage?

Solution. The ratio D_m/D_{ss} or D/r is equal to $60/0.23 = 261$; and the corresponding susceptance per mile to neutral, from table XV, page 86, is 6.07 micromhos. The charging current for the 10 miles, per phase, is $10 \times 6.07 \times 10^{-6} \times 11{,}000/\sqrt{3} = 0.39$ amp, which is obviously negligible in comparison with a load current several hundred times as great.

From table IV, d_s is found to be equal to 0.1791 in.; D_m/D_s is $60/0.1791 = 335$; and from table VII the reactance per mile is found to be 0.705 ohm. The resistance per mile is 0.265 ohm at 60 cps, from table IV.

The load or receiving-end current, which will be assumed to flow unchanged through the entire line impedance of $2.65 + j\,7.05$ ohms per phase, is

$$I = \frac{2500}{0.80 \times 11.0\sqrt{3}} = 164 \text{ amp.}$$

Following the vector diagram of figure 47, or equation (145), the line-to-line sending-end voltage E_S is

$$\begin{aligned} E_S &= \sqrt{3}\sqrt{\left\{\left(\frac{E_R}{\sqrt{3}}\cos\theta_R + IR\right)^2 + \left(\frac{E_R}{\sqrt{3}}\sin\theta_R - IX\right)^2\right\}} \\ &= \sqrt{3}\sqrt{\{(6350 \times 0.80 + 164 \times 2.65)^2} \\ &\qquad\qquad \overline{+ [6350 \times (-0.60) - 164 \times 7.05]^2\}} \\ &= \sqrt{3}\sqrt{5515^2 + 4968^2} \\ &= \sqrt{3} \times 7440 = 12{,}900 \text{ volts.} \end{aligned}$$

The phase angle between current and line-to-neutral voltage at the sending end is $\tan^{-1} 4968/5515 = 42.0$ degrees.

49. Long Lines. A line is said to be electrically long if the current that flows from wire to wire, wire to ground or wire to neutral along the line is at least a considerable fraction of the rated current of the line. According to this definition one line may be electrically longer than another, and at the same time physically shorter. The shunt current between conductors or to ground or neutral is due to the leakage conductance of the line through the insulation, and if the voltage is alternating it is due also to the capacitance. In power transmission lines the leakage current is never more than a very small fraction of 1 per cent of the full-load current, unless there is something radically wrong with the insulation. It can therefore be neglected.

The distributed capacitance may be thought of as a large number of small condensers connected between each line conductor and a hypothetical neutral conductor at short intervals, as illustrated in figure 48. If the voltage were the same all along the line from end to end, we could find the magnitude of the charging current by assuming all the condensers of figure 48 to be lumped together at any point on the line. However, the voltage varies from point to point owing to the impedance drop, and the drop per unit length also varies from point to point owing to the effect of the charging current. It is necessary therefore to set up and solve differential equations which state the fundamental current and voltage relations in an elementary section of the line.

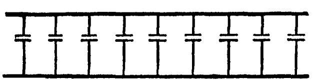

FIG. 48. Transmission line with distributed capacitance.

50. Direct-Current Case. Let us consider first the direct-current case of a long line with distributed constants, not because of its importance in power transmission, but because it facilitates the physical understanding of the phenomena. It also offers an opportunity for the student to familiarize himself with the functions involved, and the methods of handling them, when they are functions of real variables. Several times as many problems relating to direct-current lines can be worked out in a given time, as can be done relating to alternating-current lines, where functions of complex variables are involved. Even a geographically long direct-current power line would hardly have appreciable leakage current, and the electrically long direct-current lines to be considered may be visualized as long telegraph wires which carry such small line currents that the leakage current becomes very appreciable, and sometimes even a controlling factor.

The following notation will be used:

r = series resistance in ohms per unit length.
g = shunt conductance in mhos per unit length, assumed uniformly distributed.
x = distance from receiving end of line.
i = current at point x.
e = voltage at point x.
l = length of line.
E_S = sending-end voltage (at $x = l$).
I_S = sending-end current.
E_R = receiving-end voltage (at $x = 0$).
I_R = receiving-end current.
$\alpha = \sqrt{rg}$ (numeric per unit length).
$\theta = \alpha l$ (numeric).
$R = rl$ (resistance of entire line).
$G = gl$ (shunt conductance of entire line).
$R_0 = \sqrt{r/g} = \sqrt{R/G}$ ohms (called the characteristic resistance).

Choice may be exercised as to whether the line parameters should apply to the complete circuit, or one conductor at a time to neutral. It is usually better to handle the entire circuit at once for any single-phase or direct-current line; for three-phase alternating current, it will be better to work with one phase and neutral.

Figure 49 represents an elementary section of a line. The resistance per unit length of both conductors together is taken as r ohms, and so the resistance of the length dx of both conductors is equal to $r\,dx$ ohms. Likewise, the shunt conductance between conductors for length dx is equal to $g\,dx$ mhos. The current and voltage at point x are designated by the symbols i and e, and at $x + dx$ by the symbols $i + di$ and $e + de$, respectively. Making use of Ohm's and Kirchhoff's laws, we may write the following differential equations:

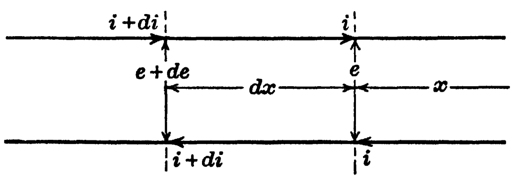

FIG. 49. Elementary section of a smooth line with uniformly distributed parameters.

$$e + de = e + ir\,dx, \quad \text{or}$$

$$\frac{de}{dx} = ir \text{ volts per unit length;} \tag{146}$$

$$i + di = i + eg\,dx,$$

or

$$\frac{di}{dx} = eg \text{ amp per unit length.} \tag{147}$$

Differentiating (146) and (147) with respect to x, we get

$$\frac{d^2e}{dx^2} = r\frac{di}{dx}, \tag{148}$$

$$\frac{d^2i}{dx^2} = g\frac{de}{dx}. \tag{149}$$

Substituting the values of $\frac{di}{dx}$ and $\frac{de}{dx}$ from (147) and (146),

$$\frac{d^2e}{dx^2} = gre = \alpha^2 e; \tag{150}$$

$$\frac{d^2i}{dx^2} = gri = \alpha^2 i. \tag{151}$$

Equations (150) and (151) state that the functions which represent the voltage and current distributions are characterized by having their second derivatives equal to α^2 times themselves. It is well known that the exponential functions $\epsilon^{\alpha x}$ and $\epsilon^{-\alpha x}$ satisfy this condition, but it is found more convenient in later numerical application to use as our solutions not the simple exponential functions, but the following combinations of them:

$$\frac{\epsilon^{\alpha x} + \epsilon^{-\alpha x}}{2} = \text{hyperbolic cosine of } \alpha x = \cosh \alpha x; \tag{152}$$

$$\frac{\epsilon^{\alpha x} - \epsilon^{-\alpha x}}{2} = \text{hyperbolic sine of } \alpha x = \sinh \alpha x. \tag{153}$$

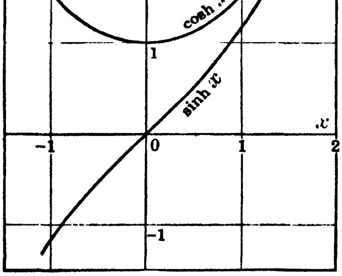

FIG. 50. Hyperbolic sines and cosines of real arguments.

The real functions of the cosh and sinh are plotted in figure 50.

We write as the solutions of (150) and (151),

$$e = A\cosh \alpha x + B\sinh \alpha x \text{ volts}, \tag{154}$$

$$i = C\cosh \alpha x + D\sinh \alpha x \text{ amp}, \tag{155}$$

in which A, B, C and D are arbitrary constants, whose values are dependent on the terminal conditions. Inserting the terminal conditions for the receiving end, at $x = 0$, we obtain:

$$E_R = A\cosh 0 + B\sinh 0 = A; \tag{156}$$

$$I_R = C\cosh 0 + D\sinh 0 = C. \tag{157}$$

The remaining undetermined constants B and D could be determined by substituting conditions at the sending end where $x = l$, but this leads to complexities, and they are most readily determined by using (146) and (147) in combination with receiving-end conditions. At $x = 0$ we have:

$$\frac{de}{dx} = I_R r \text{ volts per unit length;} \quad (158)$$

$$\frac{di}{dx} = E_R g \text{ amp per unit length.} \quad (159)$$

Differentiating (154) and (155) with respect to x, then setting $x = 0$ and substituting in (158) and (159), we have:

$$A\alpha \sinh 0 + B\alpha \cosh 0 = I_R r; \quad (160)$$

$$C\alpha \sinh 0 + D\alpha \cosh 0 = E_R g; \quad (161)$$

or

$$B = \frac{I_R r}{\alpha} = I_R R_0 \text{ volts;} \quad (162)$$

$$D = \frac{E_R g}{\alpha} = E_R/R_0 \text{ amp.} \quad (163)$$

The four arbitrary constants have all been determined, and the equations for voltage and current distribution are:

$$e = E_R \cosh \alpha x + I_R R_0 \sinh \alpha x \text{ volts;} \quad (164)$$

$$i = I_R \cosh \alpha x + \frac{E_R}{R_0} \sinh \alpha x \text{ amp.} \quad (165)$$

Setting $x = l$, which makes $\alpha x = \alpha l = \theta$, gives the expressions for sending-end current and voltage:

$$E_S = E_R \cosh \theta + I_R R_0 \sinh \theta \text{ volts;} \quad (166)$$

$$I_S = I_R \cosh \theta + \frac{E_R}{R_0} \sinh \theta \text{ amp.} \quad (167)$$

Explicit expressions for receiving-end values may be obtained by solving (166) and (167) simultaneously. We have:

$$\begin{cases} E_R \cosh \theta + I_R R_0 \sinh \theta = E_S; & (166a) \\ E_R \dfrac{\sinh \theta}{R_0} + I_R \cosh \theta = I_S; & (167a) \end{cases}$$

whence

$$E_R = \frac{\begin{vmatrix} E_S & R_0 \sinh \theta \\ I_S & \cosh \theta \end{vmatrix}}{\begin{vmatrix} \cosh \theta & R_0 \sinh \theta \\ \dfrac{\sinh \theta}{R_0} & \cosh \theta \end{vmatrix}} = \frac{E_S \cosh \theta - I_S R_0 \sinh \theta}{\cosh^2 \theta - \sinh^2 \theta = 1}. \quad (168)$$

$$I_R = \frac{\begin{vmatrix} \cosh \theta & E_S \\ \dfrac{\sinh \theta}{R_0} & I_S \end{vmatrix}}{\begin{vmatrix} \cosh \theta & R_0 \sinh \theta \\ \dfrac{\sinh \theta}{R_0} & \cosh \theta \end{vmatrix}} = I_S \cosh \theta - \frac{E_S}{R_0} \sinh \theta. \quad (169)$$

TABLE XVII

Hyperbolic Sines and Cosines

Argument	Function	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	sinh	0.0000	0.0100	0.0200	0.0300	0.0400	0.0500	0.0600	0.0701	0.0801	0.0901
	cosh	1.0000	1.0001	1.0002	1.0005	1.0008	1.0013	1.0018	1.0025	1.0032	1.0041
0.1	sinh	0.1002	0.1102	0.1203	0.1304	0.1405	0.1506	0.1607	0.1708	0.1810	0.1911
	cosh	1.0050	1.0061	1.0072	1.0085	1.0098	1.0113	1.0128	1.0145	1.0162	1.0181
0.2	sinh	0.2013	0.2115	0.2218	0.2320	0.2423	0.2526	0.2629	0.2733	0.2837	0.2941
	cosh	1.0201	1.0221	1.0243	1.0266	1.0289	1.0314	1.0340	1.0367	1.0395	1.0423
0.3	sinh	0.3045	0.3150	0.3255	0.3360	0.3466	0.3572	0.3678	0.3785	0.3892	0.4000
	cosh	1.0453	1.0484	1.0516	1.0549	1.0584	1.0619	1.0655	1.0692	1.0731	1.0770
0.4	sinh	0.4108	0.4216	0.4325	0.4434	0.4543	0.4653	0.4764	0.4875	0.4986	0.5098
	cosh	1.0811	1.0852	1.0895	1.0939	1.0984	1.1030	1.1077	1.1125	1.1174	1.1225
0.5	sinh	0.5211	0.5324	0.5438	0.5552	0.5666	0.5782	0.5897	0.6014	0.6131	0.6248
	cosh	1.1276	1.1329	1.1383	1.1438	1.1494	1.1551	1.1609	1.1669	1.1730	1.1792
0.6	sinh	0.6367	0.6485	0.6605	0.6725	0.6846	0.6967	0.7090	0.7213	0.7336	0.7461
	cosh	1.1855	1.1919	1.1984	1.2051	1.2119	1.2188	1.2258	1.2330	1.2402	1.2476
0.7	sinh	0.7586	0.7712	0.7838	0.7966	0.8094	0.8223	0.8353	0.8484	0.8615	0.8748
	cosh	1.2552	1.2628	1.2706	1.2785	1.2865	1.2947	1.3030	1.3114	1.3199	1.3286
0.8	sinh	0.8881	0.9015	0.9150	0.9286	0.9423	0.9561	0.9700	0.9840	0.9981	1.0122
	cosh	1.3374	1.3464	1.3555	1.3647	1.3740	1.3835	1.3932	1.4029	1.4128	1.4229
0.9	sinh	1.0265	1.0409	1.0554	1.0700	1.0847	1.0995	1.1144	1.1294	1.1446	1.1598
	cosh	1.4331	1.4434	1.4539	1.4645	1.4753	1.4862	1.4973	1.5085	1.5199	1.5314
1.0	sinh	1.1752	1.1907	1.2063	1.2220	1.2379	1.2539	1.2700	1.2862	1.3025	1.3190
	cosh	1.5431	1.5549	1.5669	1.5790	1.5913	1.6038	1.6164	1.6292	1.6421	1.6552
1.1	sinh	1.3356	1.3524	1.3693	1.3863	1.4035	1.4208	1.4382	1.4558	1.4735	1.4914
	cosh	1.6685	1.6820	1.6956	1.7093	1.7233	1.7374	1.7517	1.7662	1.7808	1.7956
1.2	sinh	1.5095	1.5276	1.5460	1.5645	1.5831	1.6019	1.6209	1.6400	1.6593	1.6788
	cosh	1.8107	1.8258	1.8412	1.8568	1.8725	1.8884	1.9045	1.9208	1.9373	1.9540
1.3	sinh	1.6984	1.7182	1.7381	1.7583	1.7786	1.7991	1.8198	1.8406	1.8617	1.8829
	cosh	1.9709	1.9880	2.0053	2.0028	2.0404	2.0583	2.0764	2.0947	2.1132	2.1320
1.4	sinh	1.9043	1.9259	1.9477	1.9697	1.9919	2.0143	2.0369	2.0597	2.0827	2.1059
	cosh	2.1509	2.1700	2.1894	2.2090	2.2288	2.2488	2.2691	2.2896	2.3103	2.3312

TABLE XVII — *Continued*

Hyperbolic Sines and Cosines

Argument	Function	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.5	sinh	2.1293	2.1529	2.1768	2.2008	2.2251	2.2496	2.2743	2.2993	2.3245	2.3499
	cosh	2.3524	2.3738	2.3955	2.4174	2.4395	2.4619	2.4845	2.5074	2.5305	2.5538
1.6	sinh	2.3756	2.4015	2.4276	2.4540	2.4800	2.5075	2.5346	2.5620	2.5896	2.6175
	cosh	2.5775	2.6013	2.6255	2.6499	2.6746	2.6995	2.7247	2.7502	2.7760	2.8020
1.7	sinh	2.6456	2.6740	2.7027	2.7317	2.7609	2.7904	2.8202	2.8503	2.8806	2.9112
	cosh	2.8283	2.8549	2.8818	2.9090	2.9364	2.9642	2.9922	3.0206	3.0493	3.0782
1.8	sinh	2.9422	2.9734	3.0049	3.0367	3.0689	3.1013	3.1340	3.1671	3.2005	3.2341
	cosh	3.1075	3.1371	3.1669	3.1972	3.2277	3.2585	3.2897	3.3212	3.3530	3.3852
1.9	sinh	3.2682	3.3025	3.3372	3.3722	3.4077	3.4432	3.4792	3.5156	3.5523	3.5894
	cosh	3.4177	3.4503	3.4838	3.5173	3.5512	3.5855	3.6201	3.6551	3.6904	3.7261
2.0	sinh	3.6239	3.6647	3.7028	3.7414	3.7803	3.8196	3.8593	3.8993	3.9398	3.9806
	cosh	3.7622	3.7987	3.8355	3.8727	3.9103	3.9483	3.9867	4.0255	4.0647	4.1043
2.1	sinh	4.0219	4.0635	4.1056	4.1480	4.1909	4.2342	4.2779	4.3221	4.3666	4.4117
	cosh	4.1443	4.1847	4.2256	4.2668	4.3083	4.3507	4.3932	4.4362	4.4797	4.5236
2.2	sinh	4.4571	4.5030	4.5494	4.5962	4.6434	4.6912	4.7394	4.7880	4.8372	4.8868
	cosh	4.5679	4.6127	4.6580	4.7037	4.7499	4.7966	4.8437	4.8914	4.9395	4.9881
2.3	sinh	4.9370	4.9876	5.0387	5.0903	5.1425	5.1951	5.2483	5.3020	5.3562	5.4109
	cosh	5.0372	5.0868	5.1370	5.1876	5.2388	5.2905	5.3427	5.3954	5.4487	5.5026
2.4	sinh	5.4662	5.5221	5.5785	5.6354	5.6929	5.7510	5.8097	5.8689	5.9288	5.9892
	cosh	5.5569	5.6119	5.6674	5.7235	5.7801	5.8373	5.8951	5.9535	6.0125	6.0721
2.5	sinh	6.0502	6.1118	6.1741	6.2369	6.3004	3.3645	6.4293	6.4946	6.5607	6.6274
	cosh	6.1323	6.1931	6.2545	6.3166	6.3793	6.4426	6.5066	6.5712	6.6365	6.7024
2.6	sinh	6.6947	6.7628	6.8315	6.9009	6.9709	7.0417	7.1132	7.1854	7.2583	7.3319
	cosh	6.7690	6.8363	6.9043	6.9729	7.0423	7.1123	7.1831	7.2546	7.3268	7.3998
2.7	sinh	7.4063	7.4814	7.5572	7.6338	7.7112	7.7894	7.8683	7.9480	8.0285	8.1098
	cosh	7.4735	7.5479	7.6231	7.6991	7.7558	7.8533	7.9316	8.0106	8.0905	8.1712
2.8	sinh	8.1919	8.2749	8.3586	8.4432	8.5287	8.6150	8.7021	8.7902	8.8791	8.9689
	cosh	8.2527	8.3351	8.4182	8.5022	8.5871	8.6728	8.7594	8.8469	8.9352	9.0244
2.9	sinh	9.0596	9.1512	9.2437	9.3371	9.4315	9.5268	9.6231	9.7203	9.8185	9.9177
	cosh	9.1146	9.2056	9.2976	9.3905	9.4844	9.5792	9.6749	9.7716	9.8693	9.9680

In table XVII are presented values of real hyperbolic sines and hyperbolic cosines with arguments from 0 to 2.99, by increments of 0.01. The values can be computed readily by means of the convergent series expressions:

$$\sinh \theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \cdots; \tag{170}$$

$$\cosh \theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \tag{171}$$

Example. A d-c ground-return telegraph line has a length of 100 miles, a resistance per mile of 10 ohms, and a shunt conductance to ground of 1.44×10^{-5} mho per mile. Find the steady-state voltage and current distribution on this line when the receiving end is grounded and 100 volts applied to the sending end.

Solution. We know that $E_S = 100$ volts and $E_R = 0$. From (166) we may solve for I_R:

$$I_R = \frac{E_S}{R_0 \sinh \theta} = \frac{100}{833 \sinh 1.2}$$
$$= \frac{100}{833 \times 1.509} = 0.0797 \text{ amp},$$

since $R_0 = \sqrt{r/g} = \sqrt{10/1.44 \times 10^{-5}} = 833$ ohms, and $\theta = \alpha l = 100\sqrt{10 \times 1.44 \times 10^{-5}} = 1.2$. The two distributions can now be found from (164) and (165), and the calculation of values for the quarter points is outlined in table XVIII.

TABLE XVIII

VOLTAGE AND CURRENT DISTRIBUTION

x (miles)	αx	$\sinh \alpha x$	$\cosh \alpha x$	e	i
0	0	0	1.0000	0.0	0.0797
25	0.30	0.3045	1.0453	20.2	0.0832
50	0.60	0.6366	1.1855	42.2	0.0944
75	0.90	1.0265	1.4331	67.9	0.1141
100	1.20	1.5095	1.8107	100.0	0.1444

The long-line formulas which have been developed should not be regarded as mere devices for grinding out the indicated results when the proper ingredients are supplied. They should be regarded rather as representing fundamental relationships, somewhat analogous to Ohm's and Kirchhoff's laws, which like those laws may be used in various combinations to work out a wide variety of problems.

Example. In the line of the preceding example, there is connected in series at one end a 100-volt battery and a 1000-ohm relay, and at the other end another 100-volt battery in aiding relation to the first, and a 500-ohm relay. Find the current through each relay.

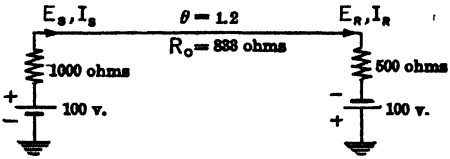

FIG. 51. Long d-c line with loads and batteries at both ends.

Solution. Referring to figure 51, and exercising due care in the matter of algebraic signs, we may proceed as follows, using I_R as the only unknown.

$$E_R = 500\, I_R - 100 \text{ volts.}$$

$$\begin{aligned} E_S &= E_R \cosh\theta + I_R R_0 \sinh\theta \\ &= (500\, I_R - 100)\, 1.8107 + I_R \times 833 \times 1.5095 \\ &= 2163\, I_R - 181.1 \text{ volts.} \end{aligned}$$

$$\begin{aligned} I_S &= I_R \cosh\theta + \frac{E_R}{R_0}\sinh\theta \\ &= 1.811\, I_R + \frac{500\, I_R - 100}{833} \times 1.5095 \\ &= 2.717\, I_R - 0.1811 \text{ amp.} \end{aligned}$$

$$\begin{aligned} 0 &= E_S + 1000\, I_S - 100 \\ &= 2163\, I_R - 181.1 + 1000(2.717\, I_R - 0.1811) - 100 \\ &= 4880\, I_R - 462.2. \end{aligned}$$

$$I_R = \frac{462.2}{4880} = 0.095 \text{ amp.}$$

$$I_S = 2.717 \times 0.095 - 0.1811 = 0.076 \text{ amp.}$$

These last two values are the currents through the relays.

Example. The line used in the two preceding examples has 100 volts impressed at the sending end. How much resistance at the receiving end is required to absorb just 1 watt of power from the line? What is the maximum power which can be drawn by a load of adjustable resistance?

Solution. In order for the load to draw 1 watt, the product $E_R I_R$ must equal unity.

$$E_S = E_R \cosh\theta + I_R R_0 \sinh\theta.$$

$$100 = \frac{1.811}{I_R} + I_R \times 833 \times 1.509.$$

Solving the quadratic equation for I_R gives:

$$I_R = 0.0517 \quad \text{or} \quad 0.0279 \text{ amp.}$$

$$E_R = 1/I_R = 19.33 \quad \text{or} \quad 35.8 \text{ volts.}$$

$$\text{Load resistance} = E_R/I_R = 374 \quad \text{or} \quad 1285 \text{ ohms.}$$

Either one of these values of load resistance will draw 1 watt from the line.

Check.

$$100 = 19.33 \times 1.811 + 0.0517 \times 833 \times 1.509$$
$$= 35.0 + 65.0. \quad \textit{Check.}$$
$$100 = 35.8 \times 1.811 + 0.0279 \times 833 \times 1.509$$
$$= 65.0 + 35.0. \quad \textit{Check.}$$

To solve for the condition of maximum receiving-end power P_R, we may write P_R/I_R in place of E_R, solve for P_R in terms of I_R, differentiate and set equal to zero for a maximum.

$$E_S = E_R \cosh \theta + I_R R_0 \sinh \theta$$
$$= \frac{P_R}{I_R} \cosh \theta + I_R R_0 \sinh \theta.$$
$$P_R = \frac{I_R (E_S - I_R R_0 \sinh \theta)}{\cosh \theta}. \tag{172}$$
$$\frac{dP_R}{dI_R} = \frac{E_S - 2\, I_R R_0 \sinh \theta}{\cosh \theta} = 0 \text{ for maximum.} \tag{173}$$
$$I_R = \frac{E_S}{2\, R_0 \sinh \theta} \text{ for maximum power,}$$
$$= \frac{100}{2 \times 833 \times 1.509} = 0.0398 \text{ amp.}$$
$$P_R = \frac{0.0398\,(100 - 0.0398 \times 833 \times 1.509)}{1.811}$$
$$= 1.10 \text{ watts maximum.}$$

The entire current-power characteristic of the line may be plotted from equation (172), and this is shown in figure 52. This curve affords

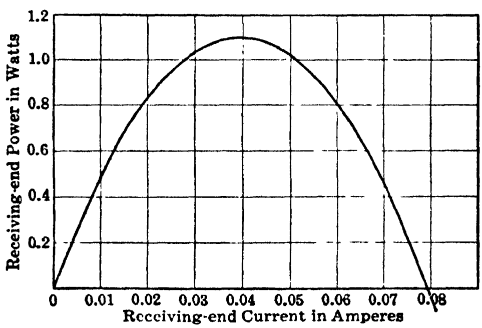

FIG. 52. Current-power characteristic of load connected to long d-c line.

a check on the work just done and also checks the solution for the short-circuit load current in the first example, since the right-hand crossing of the axis in figure 52 must correspond to a short circuit at the receiving end.

51. Alternating-Current Lines. We are accustomed to extending the use of direct-current relations of currents and voltages in circuits to cover the alternating-current case, by the use of complex impedances and admittances in place of simple resistances and conductances. In

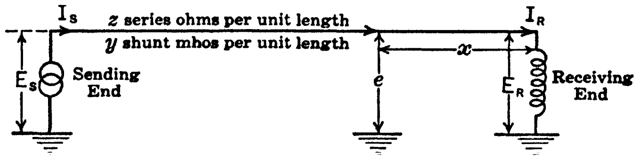

FIG. 53. Nomenclature used in the analysis of current and voltage distribution on a long line.

like manner we may make the transition to alternating currents in our long-line solution.

The same general notation as before will be used (see also figure 53), except where the introduction of vector quantities requires a change. Let:

z = series impedance per unit length, per phase.
y = shunt admittance per unit length, per phase to neutral.
x = distance from receiving end of line.
i = vector rms current at point x.
e = vector rms voltage to neutral at point x.
E_S = vector rms voltage to neutral at sending end.
I_S = vector rms current at sending end.
E_R = vector rms voltage to neutral at receiving end.
I_R = vector rms current at receiving end.
$\alpha = \sqrt{zy}$ (numeric per unit length).
$Z = zl = R + jX$ = total series impedance per phase.
$Y = yl = G + jB$ = total shunt admittance per phase to neutral.
$\theta = \alpha l = \sqrt{ZY}$ (numeric).
$Z_0 = \sqrt{z/y} = \sqrt{Z/Y}$ ohms (called the characteristic impedance).

Proceeding as before in setting up and solving the differential equations, we have:

$$\frac{de}{dx} = iz \text{ volts per unit length,} \tag{174}$$

$$\frac{di}{dx} = ey \text{ amp per unit length.} \tag{175}$$

Eliminating i and e in turn,

$$\frac{d^2e}{dx^2} = \alpha^2 e, \tag{176}$$

$$\frac{d^2i}{dx^2} = \alpha^2 i. \tag{177}$$

The solutions, with terminal conditions brought in as before, are:

$$e = E_R \cosh \alpha x + I_R Z_0 \sinh \alpha x, \tag{178}$$

$$i = I_R \cosh \alpha x + \frac{E_R}{Z_0} \sinh \alpha x. \tag{179}$$

It is somewhat more convenient to substitute Z/θ for Z_0 and Y/θ for $1/Z_0$ in writing the connections between terminal voltages and currents.

$$E_S = E_R \cosh \theta + I_R Z \frac{\sinh \theta}{\theta} \text{ volts}, \tag{180}$$

$$I_S = I_R \cosh \theta + E_R Y \frac{\sinh \theta}{\theta} \text{ amp.} \tag{181}$$

$$E_R = E_S \cosh \theta - I_S Z \frac{\sinh \theta}{\theta} \text{ volts}, \tag{182}$$

$$I_R = I_S \cosh \theta - E_S Y \frac{\sinh \theta}{\theta} \text{ amp.} \tag{183}$$

These hyperbolic functions have as their arguments the numeric θ, which is complex, containing both a real and imaginary term. The exponential and series expressions are still valid, however. For example, to compute the hyperbolic cosine of $1.0\underline{|60^\circ} = \cosh\left(0.5 + j\frac{\sqrt{3}}{2}\right)$ we may use the series as follows:

$$\cosh \theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots.$$

$$\begin{aligned}\cosh 1\underline{|60^\circ} &= 1 + \frac{1\underline{|120^\circ}}{2} + \frac{1\underline{|240^\circ}}{24} + \frac{1\underline{|360^\circ}}{720} + \cdots \\ &= (1 - 0.250 - 0.021 + 0.001 + \cdots) \\ &\qquad + j(0.433 - 0.036 + 0 + \cdots) \\ &= 0.730 \cdots + j\,0.397 \cdots \\ &= 0.831\underline{|28.5^\circ}.\end{aligned}$$

Charts I and II furnish rapid means of determining the complex values of $\cosh \theta$ and $\frac{\sinh \theta}{\theta}$. These charts cover a range of 0 to 290 miles for line length at 60 cycles per second, and proportionately more for lower frequencies. As a convenience, the independent variable used in the mapping is ZY, or θ^2, rather than θ. This obviates the necessity of taking the square root of ZY to find θ, and has the further advantage of causing adjacent curves of constant size of the independent variable to be nearly equidistant, which is a considerable advantage in legible mapping and accurate reading.

By careful use of the charts, a precision of 1 part in 10,000 may be obtained. If only slide-rule accuracy is desired, it may be obtained by reading only to the nearest divisions of the function scales.

Example of use of charts. A three-phase transmission line has total series impedance per phase Z of $200.0\,\underline{|80.00^\circ}$ vector ohms, and total shunt admittance Y of $0.00130\,\underline{|90.00^\circ}$ vector mho per phase to neutral. Find $\cosh\theta$ and $\dfrac{\sinh\theta}{\theta}$.

$$ZY = 200.0\,\underline{|80.00^\circ} \times 0.00130\,\underline{|90.00^\circ} = 0.260\,\underline{|170.00^\circ}.$$

Entering chart I with this value of ZY, it is found that $\cosh\theta = 0.8750\,\underline{|1.41^\circ}$. Similarly, from chart II, $\dfrac{\sinh\theta}{\theta} = 0.9578\,\underline{|0.44^\circ}$.

Example. Current and voltage solution. The line just described delivers to a balanced load of 0.90 power factor, lagging current, a total power of 90,000 kw, the receiving-end voltage being 200,000 between lines. Find (*a*) the sending-end voltage, (*b*) sending-end current, (*c*) sending-end power factor, (*d*) efficiency of transmission, and (*e*) the value to which the receiving-end voltage would rise upon removal of load if the sending-end voltage is unchanged.

Solution. (***a***). The voltage to neutral at the receiving end, E_R, is $200{,}000/\sqrt{3} = 115{,}500$. Let this be taken as the axis of reference. The receiving-end current per phase is

$$\begin{aligned}
I_R &= \frac{90{,}000}{0.90 \times 200\sqrt{3}}\,\underline{|-\cos^{-1} 0.90} \\
&= 289\,\underline{|-25.84^\circ} \text{ amp.} \\
E_S &= E_R \cosh\theta + I_R Z \frac{\sinh\theta}{\theta} \\
&= 115{,}500\,\underline{|0^\circ} \times 0.875\,\underline{|1.41^\circ} + 289\,\underline{|-25.84^\circ} \times 200\,\underline{|80.0^\circ} \times 0.958\,\underline{|0.44^\circ} \\
&= 101{,}100\,\underline{|1.41^\circ} + 55{,}400\,\underline{|54.60^\circ} \\
&= 101{,}100 + j\,2500 + 32{,}100 + j\,45{,}200 \\
&= 133{,}200 + j\,47{,}700 \\
&= 141{,}700\,\underline{|19.7^\circ} \text{ volts to neutral.}
\end{aligned}$$

This corresponds to 245,000 volts between lines,

$$
\begin{aligned}
(b)\ I_S &= I_R \cosh\theta + E_R Y \frac{\sinh\theta}{\theta} \\
&= 289\,\underline{|-25.84^\circ} \times 0.875\,\underline{|1.41^\circ} + 115{,}500\,\underline{|0^\circ} \\
&\qquad\qquad \times 0.00130\,\underline{|90.00^\circ} \times 0.958\,\underline{|0.44^\circ} \\
&= 252.9\,\underline{|-24.43^\circ} + 143.8\,\underline{|90.44^\circ} \\
&= 230 - j\,104.3 - 1.1 + j\,143.8 \\
&= 229 + j\,39.5 \\
&= 232\,\underline{|9.8^\circ}\ \text{amp.}
\end{aligned}
$$

(*c*) The sending-end power factor is the cosine of the phase angle between E_S and I_S. This angle is 19.7 − 9.8 = 9.9 degrees, with the current lagging, and the power factor at the sending end is therefore 0.985.

(*d*) The efficiency η is the ratio of receiving-end power to sending-end power.

$$
\eta = \frac{30{,}000}{141.7 \times 232 \times 0.985} = \frac{30{,}000}{32{,}400} = 0.926.
$$

(*e*) If the load is removed and E_S remains 141,700 volts to neutral, we may use equation (180), in which I_R is now zero, in order to find E_R.

$$
\begin{aligned}
E_R &= \frac{E_S}{\cosh\theta} = \frac{141{,}700\,\underline{|19.7^\circ}}{0.875\,\underline{|1.41^\circ}} \\
&= 161{,}800\,\underline{|18.3^\circ}\ \text{volts to neutral.}
\end{aligned}
$$

This corresponds to a line-to-line voltage of 280,000, and the receiving-end voltage is advanced 18.3 degrees in phase ahead of its former position under load.

At first it is a little startling to find that the receiving-end voltage may exceed considerably the sending-end voltage, but that is the fact. The reason is that at no load or at light loads the line becomes in effect a series circuit whose principal impedances are the series inductive reactance and the capacitive reactance to neutral which is usually handled as a shunt susceptance. This series circuit is in partial resonance.

52. Further Examples. The transmission engineer may need to solve many problems for which the formulas developed do not supply direct explicit solutions. One such problem is that of determining the receiving-end voltage when the sending-end voltage and the receiving-end vector power are known.

Example. In the line described in the preceding article, the load receives 60,000 kw at 0.80 power factor, lagging current, and the sending-end voltage is 220,000 between lines. Find the receiving-end voltage.

Solution. The receiving-end voltage to neutral, E_R, will be taken as the axis of reference. Then the receiving-end current must be

$$I_R = \frac{20{,}000{,}000}{0.80\,E_R}\underline{\lfloor -36.9^\circ} \text{ amp.}$$

Using equation (180), we can write

$$E_S = E_R \cosh\theta + I_R Z \frac{\sinh\theta}{\theta}$$

$$= E_R \times 0.875\underline{\lfloor 1.41^\circ} + \frac{25{,}000{,}000}{E_R}\underline{\lfloor -36.9^\circ} \times 200\underline{\lfloor 80.0^\circ} \times 0.958\underline{\lfloor 0.44^\circ}$$

$$= 0.875\underline{\lfloor 1.41^\circ}\,E_R + \frac{4{,}790{,}000{,}000}{E_R}\underline{\lfloor 43.5^\circ} \text{ volts.}$$

Now E_S in this equation is known to have a size of $220{,}000/\sqrt{3} = 127{,}000$ volts, but, since the reference axis has been selected along the vector E_R, there is an unknown phase angle associated with E_S. However, the right-hand side of the equation can be completely separated into its real and imaginary components, and the sum of the squares of these two components equated to the square of the known size of E_S.

$$127{,}000\underline{\lfloor ?} = 0.875E_R + j0.02155E_R + \frac{3{,}480{,}000{,}000}{E_R} + j\frac{3{,}300{,}000{,}000}{E_R}.$$

$$127{,}000^2 = \left(0.875\,E_R + \frac{3{,}480{,}000{,}000}{E_R}\right)^2 + \left(0.02155\,E_R + \frac{3{,}300{,}000{,}000}{E_R}\right)^2.$$

Reducing,

$$0.7656\,E_R^4 - 9.90 \times 10^9\,E_R^2 + 22.94 \times 10^{18} = 0.$$

Solving by the quadratic formula,

$$E_R^2 = 99.0 \times 10^8 \quad \text{or} \quad 30.2 \times 10^8;$$

$$E_R = 99{,}500 \quad \text{or} \quad 55{,}000 \text{ volts to neutral.}$$

Either of these voltages represents a possible mathematical solution, but as a practical matter the higher voltage should be selected because the other represents a condition approaching short circuit, with very poor efficiency and voltage regulation, and very probable overheating. The line voltage will be 172,300.

Check. With E_R equal to 99,500 $\underline{|0^\circ}$,

$$I_R = 20{,}000\ \underline{|-36.9^\circ}/0.80 \times 99.5\ \underline{|0^\circ} = 251.3\ \underline{|-36.9^\circ} \text{ amp.}$$

$$E_S = 99{,}500\ \underline{|0^\circ} \times 0.875\ \underline{|1.41^\circ} + 251.3\ \underline{|-36.9^\circ} \times 200\ \underline{|80.0^\circ} \times 0.958\ \underline{|0.44^\circ}$$
$$= 87{,}100\ \underline{|1.41^\circ} + 48{,}150\ \underline{|43.5^\circ}$$
$$= 87{,}100 + j\,2140 + 34{,}900 + j\,33{,}100$$
$$= 122{,}000 + j\,35{,}200 = 127{,}000\ \underline{|16.1^\circ} \text{ volts to neutral.}$$

This checks the original datum.

Example. Two long smooth lines are in parallel. Determine the sending-end voltage in terms of receiving-end current and voltage. There is no electric or magnetic coupling between the lines.

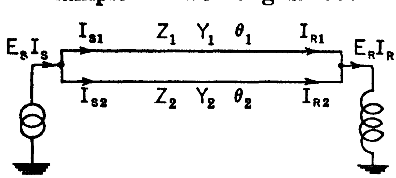

FIG. 54. Two long lines in parallel.

Solution. Referring to figure 54, and using the notation indicated there, we may write the hyperbolic function formulas for each line, connecting them by the relations that $I_{S_1} + I_{S_2} = I_S$, and $I_{R_1} + I_{R_2} = I_R$.

$$E_S = E_R \cosh\theta_1 + I_{R_1} Z_1 \frac{\sinh\theta_1}{\theta_1}, \tag{184}$$

$$E_S = E_R \cosh\theta_2 + (I_R - I_{R_1})\, Z_2 \frac{\sinh\theta_2}{\theta_2}. \tag{185}$$

Solving simultaneously for I_{R_1}, substituting this back into (184) and simplifying, gives the result:

$$E_S = E_R \left[\frac{Z_1 \dfrac{\sinh\theta_1}{\theta_1}\cosh\theta_2 + Z_2 \dfrac{\sinh\theta_2}{\theta_2}\cosh\theta_1}{Z_1 \dfrac{\sinh\theta_1}{\theta_1} + Z_2 \dfrac{\sinh\theta_2}{\theta_2}} \right] + I_R \left[\frac{Z_1 Z_2 \dfrac{\sinh\theta_1}{\theta_1}\dfrac{\sinh\theta_2}{\theta_2}}{Z_1 \dfrac{\sinh\theta_1}{\theta_1} + Z_2 \dfrac{\sinh\theta_2}{\theta_2}} \right] \text{volts.} \tag{186}$$

For the special case where the two lines are identical, $Z_1 = Z_2 = Z$, and $\theta_1 = \theta_2 = \theta$, and equation (186) reduces to

$$E_S = E_R \cosh\theta + I_R \frac{Z}{2}\frac{\sinh\theta}{\theta}. \tag{187}$$

This result is the same that would have been obtained by combining the constants of the two identical lines, using 2 Y for the combined admittance and $\frac{Z}{2}$ for the combined parallel impedance. In the general case, however, when the line constants are not identical, it is not correct to use an impedance equal to that of the two individual impedances in parallel, and an admittance equal to the sum of the individual admittances.

Example. The line previously described ($Z = 200.0 \underline{|80.0°}$, $Y = 0.00130 \underline{|90.0°}$) has a sending-end voltage of 220,000 line-to-line, and the total load consists of synchronous reactors operating at 2.0 per cent power factor in parallel with a balanced load of 90,000 kw at 0.90 power factor, lagging current. The load voltage is maintained at 200,000 between lines by means of field control of the reactors. Find the number of kilovolt amperes taken by the reactors.

Solution. Let the reactor line current per phase be $I_C \underline{|\cos^{-1} 0.02} = 0.02\, I_C + j\, 1.00\, I_C$, on the assumption that this current will be leading. The receiving-end voltage is taken as the axis of reference. The current taken by the load exclusive of the reactors is

$$90{,}000 \underline{|-\cos^{-1} 0.90} \;/\; (0.90 \times 200\sqrt{3}) = 289 \underline{|-25.84°}$$
$$= 260 - j\,126 \text{ amp.}$$

The total load current is the sum of this and the reactor current, or $(260 + 0.02\, I_C) + j\,(-126 + I_C) \text{ amp} = I_R$.

$$E_S = E_R \cosh\theta + I_R Z \frac{\sinh\theta}{\theta}$$
$$= 115{,}500 \times 0.875 \underline{|1.41} + [(260 + 0.02\, I_C) + j\,(-126 + I_C)]$$
$$200 \underline{|80.0°} \times 0.958 \underline{|0.44}$$
$$= 101{,}100 + j\,2500 + (32{,}100 - 192.7\, I_C) + j\,(45{,}100 + 32.4\, I_C)$$
$$= (133{,}200 - 192.7\, I_C) + j\,(47{,}600 + 32.4\, I_C) \text{ volts.}$$

The phase angle of E_S with respect to the reference axis is unknown, but the square of the size of E_S must equal the sum of the squares of the real and imaginary components on the right of the equality sign.

$$127{,}000^2 = (133{,}200 - 192.7\, I_C)^2 + (47{,}600 + 32.4\, I_C)^2.$$

Solving, $I_C = +86.3$ or $+1177$ amp. The smaller value is the practical one, and the kilovolt amperes taken by the reactors will be $86.3 \times 200\sqrt{3} = 29{,}900$. The positive sign for I_C indicates leading current, that being the original assumption.

If the solution had indicated a negative value of I_C it would have been necessary to do the work over again based on the starting assumption of a reactor current equal to $0.02\, I_C - j\, 1.00\, I_C$. It is to be noted that the reactive component may be of either sign, but the real or in-phase component must always be positive.

Check. Based on the result obtained, the total receiving-end current is

$$\begin{aligned} I_R &= 260 - j\,126 + 1.7 + j\,86.3 \\ &= 262 - j\,39.7 = 265 \angle{-8.61^\circ} \text{ amp.} \\ E_S &= 115{,}500 \angle 0^\circ \times 0.875 \angle 1.41^\circ + 265 \angle{-8.61^\circ} \times 200 \angle 80.0^\circ \\ &\qquad \times 0.958 \angle 0.44^\circ \\ &= 101{,}100 + j\,2500 + 15{,}800 + j\,48{,}100 \\ &= 127{,}000 \angle 23.4^\circ \text{ volts. } \textit{Check.} \end{aligned}$$

53. Equivalent π and T Lines. For experimental model representation of long lines in the steady state at rated frequency, and sometimes for computation, it is convenient to use a lumped-constant system which is equivalent, in its characteristics between terminals, to the actual smooth line.

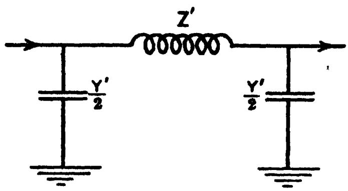

FIG. 55. Equivalent π of a smooth line.

Referring to figure 55, let it be assumed that the connection shown can represent exactly the smooth line, in the steady state and at the normal frequency. It is desired to determine the values which Z' and Y' must have. Using well-known methods, the sending-end current and voltage can be expressed in terms of the receiving-end current and voltage as follows:

$$E_S = E_R\left(1 + \frac{Z'Y'}{2}\right) + I_R Z'; \tag{188}$$

$$I_S = I_R\left(1 + \frac{Z'Y'}{2}\right) + E_R\left(Y' + \frac{Z'Y'^2}{4}\right). \tag{189}$$

By the smooth-line formulas, equations (180) and (181), these can be expressed as follows:

$$E_S = E_R \cosh\theta + I_R Z \frac{\sinh\theta}{\theta} \text{ volts;} \tag{190}$$

$$I_S = I_R \cosh\theta + E_R Y \frac{\sinh\theta}{\theta} \text{ amp.} \tag{191}$$

In order for the connection of figure 55 to be the exact equivalent of the original line having constants Z and Y, equation (188) must be identical in value with (190), and (189) identical with (191). Since E_R and I_R may be varied independently of each other, it follows that terms having a factor E_R in (188) and (189) must be equal to terms having a factor E_R in (190) and (191) respectively; and likewise, the same holds true for terms having a factor I_R. Therefore, from the last terms of (188) and (190),

$$Z' = Z\frac{\sinh\theta}{\theta}\text{ vector ohms.} \tag{192}$$

From the first terms after the equality sign in these same equations,

$$\cosh\theta = 1 + \frac{Z'Y'}{2} = 1 + \frac{Y'}{2}\frac{Z\sinh\theta}{\theta}. \tag{193}$$

$$\frac{Y'}{2} = \frac{\theta(\cosh\theta - 1)}{Z\sinh\theta} = \frac{\sqrt{ZY}(\cosh\theta - 1)}{Z\sinh\theta}$$

$$= \frac{Y(\cosh\theta - 1)}{\theta\sinh\theta} = \frac{Y}{2}\frac{\tanh\theta/2}{\theta/2}\text{ vector mhos.} \tag{194}$$

The fact that $\dfrac{\cosh\theta - 1}{\sinh\theta} = \tanh\theta/2$ can easily be proved by using the exponential forms.

$$\frac{\cosh\theta - 1}{\sinh\theta} = \frac{\epsilon^{\theta} + \epsilon^{-\theta} - 2}{\epsilon^{\theta} - \epsilon^{-\theta}} = \frac{(\epsilon^{\theta} - 2 + \epsilon^{-\theta}) \div (\epsilon^{\theta/2} - \epsilon^{-\theta/2})}{(\epsilon^{\theta} - \epsilon^{-\theta}) \div (\epsilon^{\theta/2} - \epsilon^{-\theta/2})}$$

$$= \frac{\epsilon^{\theta/2} - \epsilon^{-\theta/2}}{\epsilon^{\theta/2} + \epsilon^{-\theta/2}} = \frac{\sinh\theta/2}{\cosh\theta/2} = \tanh\theta/2. \tag{195}$$

To complete the proof we must examine also the terms in (189) and (191) containing E_R, and see if they are made equal to each other by the same correction factors. We must show that

$$Y' + \frac{Z'Y'^2}{4} = Y\frac{\sinh\theta}{\theta} \tag{196}$$

when $Y' = Y\dfrac{\tanh\theta/2}{\theta/2}$ and $Z' = Z\dfrac{\sinh\theta}{\theta}$. Making these substitutions, and writing for $\tanh\theta/2$ its value as given in (195),

$$Y\frac{\cosh\theta - 1}{\frac{\theta}{2}\sinh\theta} + \frac{ZY^2(\cosh\theta - 1)^2}{\theta^3\sinh\theta} = Y\frac{\sinh\theta}{\theta}. \tag{197}$$

$$2\cosh\theta - 2 + \cosh^2\theta - 2\cosh\theta + 1 = \sinh^2\theta. \tag{198}$$

The cosh θ terms cancel, and since $\cosh^2\theta - \sinh^2\theta = 1$, the equation

We have therefore:

$$Z' = Z\frac{\sinh\theta}{\theta}\text{ ohms in `` architrave '';} \tag{199}$$

$$\frac{Y'}{2} = \frac{Y}{2}\frac{\tanh\theta/2}{\theta/2}\text{ mhos in each `` pillar.''} \tag{200}$$

In a similar manner, the correction factors for the equivalent T line may be worked out. The formulas are (see figure 56)

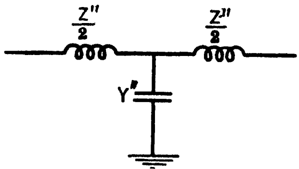

FIG. 56. Equivalent T of a smooth line.

$$Y'' = Y\frac{\sinh\theta}{\theta}\text{mhos,} \tag{201}$$

$$\frac{Z''}{2} = \frac{Z}{2}\frac{\tanh\theta/2}{\theta/2}\text{ ohms.} \tag{202}$$

Hyperbolic function charts II and III present values of the correction factors needed for determining the constants of both equivalent π and equivalent T lines.

Example. Find the constants of the equivalent π of a smooth line whose total series impedance per phase is $180\,\underline{|75.0^\circ}$ ohms, and whose shunt admittance to neutral per phase is $0.00100\,\underline{|90.0^\circ}$ mho.

Solution. $ZY = 180\,\underline{|75.0^\circ} \times 0.00100\,\underline{|90.0^\circ} = 0.180\,\underline{|165.0^\circ}.$ From chart II, $\frac{\sinh\theta}{\theta} = 0.971\,\underline{|0.45^\circ}$. From chart III, $\frac{\tanh\theta/2}{\theta/2} = 1.015\,\underline{|-0.23^\circ}$. $Z' = 180\,\underline{|75.0^\circ} \times 0.971\,\underline{|0.45^\circ} = 174.8\,\underline{|75.45^\circ}$ ohms. $Y' = 0.00100\,\underline{|90.0^\circ} \times 1.015\,\underline{|-0.23^\circ} = 0.001015\,\underline{|89.77^\circ}$ mho.

A peculiarity of the use of the equivalent T is that the admittance of the pillar or leg of the T usually turns out to have an angle greater than 90 degrees, which would indicate a negative resistance. For this reason it is not practicable to construct models of equivalent T sections in most cases. The analytic solution based on the T is valid, however. The actual ohmic line loss is not necessarily equal to the architrave loss of its equivalent π under the same loading conditions. The equivalence applies merely to the terminal characteristics and does not extend to the corresponding component parts.

For short power lines, the values of the two hyperbolic correction factors $\frac{\sinh\theta}{\theta}$ and $\frac{\tanh\theta/2}{\theta/2}$ both approach $1\,\underline{|0^\circ}$, and a π or T line in which this value has been used as an approximation is sometimes called a " nominal " π or T.

In addition to its usefulness in constructing models, the equivalent line, usually of the π type, is almost indispensable in reducing and simplifying networks of long lines for the computation of short-circuit or other currents.

54. Other Methods of Computing Complex Hyperbolic Functions. Unavailability of charts such as those in this book, or the need of determining functions beyond the range of the charts, may at times require the use of other methods of computing.

1. **From series.** Direct computation may be made from the series, as has already been stated. If series are to be used, it is best to use the following formulas:

$$E_S = E_R\left(1 + \frac{ZY}{2} + \frac{Z^2Y^2}{24} + \frac{Z^3Y^3}{720} + \cdots\right) + I_R Z\left(1 + \frac{ZY}{6} + \frac{Z^2Y^2}{120} + \frac{Z^3Y^3}{5040} + \cdots\right) \text{ volts}; \tag{203}$$

$$I_S = I_R\left(1 + \frac{ZY}{2} + \frac{Z^2Y^2}{24} + \frac{Z^3Y^3}{720} + \cdots\right) + E_R Y\left(1 + \frac{ZY}{6} + \frac{Z^2Y^2}{120} + \frac{Z^3Y^3}{5040} + \cdots\right) \text{ amp.} \tag{204}$$

$$\frac{\tanh \theta/2}{\theta/2} = 1 - \frac{ZY}{12} + \frac{Z^2Y^2}{120} - \frac{17\, Z^3Y^3}{20{,}160} + \cdots. \tag{205}$$

2. **From expansions in terms of functions of real variables.** To expand $\cosh\,(u + jv)$ in terms of functions of u alone and v alone, the exponential form can be used.

$$\begin{aligned}\cosh\,(u + jv) &= \tfrac{1}{2}\,(\epsilon^{u+jv} + \epsilon^{-u-jv}) \\ &= \tfrac{1}{4}\,(\epsilon^{u+jv} + \epsilon^{-u+jv} + \epsilon^{u-jv} + \epsilon^{-u-jv}) + \tfrac{1}{4}\,(\epsilon^{u+jv} - \epsilon^{-u+jv} - \epsilon^{u-jv} + \epsilon^{-u-jv} \\ &= \frac{\epsilon^u + \epsilon^{-u}}{2}\cdot\frac{\epsilon^{jv} + \epsilon^{-jv}}{2} + j\,\frac{\epsilon^u - \epsilon^{-u}}{2}\cdot\frac{\epsilon^{jv} - \epsilon^{-jv}}{2j} \\ &= \cosh u \cos v + j \sinh u \sin v.\end{aligned} \tag{206}$$

In a similar manner other expansions can be obtained. The most useful are:

$$\sinh\,(u + jv) = \sinh u \cos v + j \cosh u \sin v \tag{207}$$

$$\tanh^{-1}\,(u \pm jv) = \tfrac{1}{4}\ln\frac{(1+u)^2 + v^2}{(1-u)^2 + v^2} + j\,\tfrac{1}{2}\left\{\pi - \tan^{-1}\left(\frac{u+1}{\pm v}\right) + \tan^{-1}\left(\frac{u-1}{\pm v}\right)\right\} \tag{208}$$

In applying formula (208), any values of the antitangents may be used, but both should be taken from the *same branch* of the tangent graph.

Circular and hyperbolic tables of real functions are available which are much more accurate than existing tables of complex functions, and so more accuracy in the final result can be obtained by using the above expansion formulas in conjunction with these tables. The labor involved is not a great deal more than that required in interpolating in the complex tables, especially since interpolation with the real tables may be completely avoided by using tables having small intervals.

3. **Kennelly's charts and tables.** Dr. A. E. Kennelly has published an extensive set of complex hyperbolic function tables, and a book of charts based on the same data as his tables. These cover a considerably wider range than the three small power-transmission-range charts included in this book.

55. Formulas for Reference. It is convenient to have collected for ready reference the various long-line formulas which have been developed.

For a loaded line:

$$E_S = E_R \cosh \theta + I_R Z \frac{\sinh \theta}{\theta}, \tag{209}$$

$$I_S = I_R \cosh \theta + E_R Y \frac{\sinh \theta}{\theta}, \tag{210}$$

$$E_R = E_S \cosh \theta - I_S Z \frac{\sinh \theta}{\theta}, \tag{211}$$

$$I_R = I_S \cosh \theta - E_S Y \frac{\sinh \theta}{\theta}, \tag{212}$$

$$e = E_R \cosh \alpha x + I_R z x \frac{\sinh \alpha x}{\alpha x}, \tag{213}$$

$$= E_S \cosh \alpha (l - x) - I_S z (l - x) \frac{\sinh \alpha (l - x)}{\alpha (l - x)}, \tag{214}$$

$$i = I_R \cosh \alpha x + E_R y x \frac{\sinh \alpha x}{\alpha x}, \tag{215}$$

$$= I_S \cosh \alpha (l - x) - E_S y (l - x) \frac{\sinh \alpha (l - x)}{\alpha (l - x)}. \tag{216}$$

For a line short-circuited at the receiving end:

$$E_R = 0, \tag{217}$$

$$I_R = \frac{I_S}{\cosh \theta} = \frac{E_S}{Z \sinh \theta / \theta}, \tag{218}$$

$$E_S = I_R Z \sinh \theta / \theta = I_S Z \tanh \theta / \theta, \tag{219}$$

$$I_S = I_R \cosh \theta = \frac{E_S}{Z \tanh \theta / \theta}, \tag{220}$$

$$e = I_R z x \sinh \alpha x/\alpha x = E_S \frac{\sinh \alpha x}{\sinh \theta} = I_S z x \frac{\sinh \alpha x/\alpha x}{\cosh \theta}, \tag{221}$$

$$i = I_R \cosh \alpha x = I_S \frac{\cosh \alpha x}{\cosh \theta} = \frac{E_S}{Z} \frac{\cosh \alpha x}{\sinh \theta/\theta}. \tag{222}$$

For a line open at the receiving end:

$$I_R = 0, \tag{223}$$

$$E_R = \frac{E_S}{\cosh \theta} = \frac{I_S Z_0}{\sinh \theta} = \frac{I_S}{Y \sinh \theta/\theta}, \tag{224}$$

$$E_S = \frac{I_S}{Y \tanh \theta/\theta} = E_R \cosh \theta, \tag{225}$$

$$I_S = E_S Y \tanh \theta/\theta = E_R Y \sinh \theta/\theta, \tag{226}$$

$$e = E_R \cosh \alpha x = E_S \frac{\cosh \alpha x}{\cosh \theta} = \frac{I_S \cosh \alpha x}{Y \sinh \theta/\theta}, \tag{227}$$

$$i = E_S y x \frac{\sinh \alpha x/\alpha x}{\cosh \theta} = E_R y x \sinh \alpha x/\alpha x = I_S \frac{\sinh \alpha x}{\sinh \theta}. \tag{228}$$

For the equivalent π:

$$Z' = Z \frac{\sinh \theta}{\theta}, \tag{229}$$

$$\frac{Y'}{2} = \frac{Y}{2} \frac{\tanh \theta/2}{\theta/2}. \tag{230}$$

For the equivalent T:

$$\frac{Z''}{2} = \frac{Z}{2} \frac{\tanh \theta/2}{\theta/2}, \tag{231}$$

$$Y'' = Y \frac{\sinh \theta}{\theta}. \tag{232}$$

56. Relations among the Hyperbolic Functions. The following relations among hyperbolic functions may be found useful in transmission work.

$$\sinh x = \frac{\epsilon^x - \epsilon^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots. \tag{233}$$

$$\cosh x = \frac{\epsilon^x + \epsilon^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots. \tag{234}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = x - \frac{x^3}{3} + \frac{2\,x^5}{15} - \frac{17\,x^7}{315} + \frac{62\,x^9}{2835} - \cdots. \tag{235}$$

$$\cosh^2 x - \sinh^2 x = 1. \tag{236}$$

$$\sinh 2\,x = 2 \sinh x \cosh x. \tag{237}$$

$$\cosh 2\,x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1. \tag{238}$$

$$\frac{d}{dx} \sinh x = \cosh x. \tag{239}$$

$$\frac{d}{dx} \cosh x = \sinh x. \tag{240}$$

$$\int \sinh x\, dx = \cosh x + \text{constant}. \tag{241}$$

$$\int \cosh x\, dx = \sinh x + \text{constant}. \tag{242}$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1. \tag{243}$$

$$\sinh\,(a \pm jb) = \sinh a \cos b \pm j \cosh a \sin b. \tag{244}$$

$$\cosh\,(a \pm jb) = \cosh a \cos b \pm j \sinh a \sin b. \tag{245}$$

$$\sinh\,(a + jb \pm j\pi) = -\sinh\,(a + jb). \tag{246}$$

$$\cosh\,(a + jb \pm j\pi = -\cosh\,(a + jb). \tag{247}$$

$$\tanh\,(a + jb \pm j\pi) = \tanh\,(a + jb). \tag{248}$$

Equations 246–8, inclusive, make it possible to extend the limit of usefulness of a given table or chart, as do also equations (237) and (238).

$$\cosh x = \cos jx. \tag{248a}$$

$$\sinh x = -j \sin jx. \tag{248b}$$

REFERENCES

A. E. Kennelly, *Hyperbolic Functions Applied to Electrical Engineering.*
Tables of Complex Hyperbolic and Circular Functions.
Chart Atlas of Complex Hyperbolic and Circular Functions.

L. F. Woodruff, *Complex Hyperbolic Function Charts*, Electrical Engineering, May, 1935.

PROBLEMS ON CHAPTER V

Prob. 1–5. Show, by integrating the expression for line voltage drop per unit length over the entire length of a loaded long line, that the total drop is equal to the difference between E_S and E_R.

Prob. 2–5. Show, by integrating the expression for the shunt current per unit length over the entire length of a loaded long line, that the total is equal to the difference between I_S and I_R.

Prob. 3–5. A d-c telegraph line has resistance of 10 ohms per mile and leakance 10^{-6} mho per mile. How much current will flow through a 500-ohm relay at one end when there is a 100-volt battery in series with the line at the other end? $l = 100$ mi.

Prob. 4–5. A 175-mile d-c telegraph line has three 500-ohm relays in series with it, one at each end and one 75 miles from one end. The constants per mile of line are $r = 10$ ohms and $g = 10^{-6}$ mho. Find the current which will flow through each relay if a 100-volt battery in series at the intermediate relay station is used to energize the line.

Prob. 5–5. Calculate and plot a curve between receiving-end resistance and power for the d-c line described in the first example of article 50. The sending-end voltage is 100. Use the following values of resistance: 0, 100, 500, 1000, 5000 and 10,000 ohms.

Prob. 6–5. A three-phase 60-cps 200-mile line has three 650,000-cir-mil standard hard-drawn stranded copper conductors, arranged in a horizontal plane with 30 ft between adjacent conductors. The receiving-end voltage is 200,000 between lines. The load is 60,000 kw at 0.90 power factor, lagging current. Find (*a*) the sending-end voltage; (*b*) the sending-end current; (*c*) the sending-end power factor; and (*d*) the efficiency of transmission.

Prob. 7–5. The line of problem 6–5 has a sending-end voltage of 220,000 between lines. Calculate and plot a curve between receiving-end voltage and power for a receiving-end power factor of (*a*) unity, (*b*) 0.90 lagging current; (*c*) 0.90 leading; (*d*) 0.80 lagging; (*e*) 0.70 lagging. (NOTE TO INSTRUCTOR: as this is a long problem it is suggested that it be divided among the class.)

Prob. 8–5. Two long smooth lines are connected in series. The total constants of the one connected to the sending end are Z_1 and Y_1, and of the other Z_2 and Y_2. Determine the sending-end voltage in terms of the receiving-end voltage and current.

Prob. 9–5. Determine the constants of the equivalent π of the line described in problem 6–5.

Prob. 10–5. Derive the formulas for the hyperbolic correction factors for determining the constants of the equivalent T line.

Prob. 11–5. A long 60-cps a-c line has the following constants: $z = 0.80 \underline{|80^\circ}$ ohm per mile; $y = 5.0 \underline{|90^\circ}$ micromhos per mile. The receiving end is open and has unit voltage. Calculate and plot the voltage distribution for a complete wave length; that is, continue working back from the receiving end until the phase angle of the voltage has shifted 360 degrees.

Prob. 12–5. Plot the current distribution for the line of problem 11–5, for a complete wave length.

Prob. 13–5. Find the receiving-end voltage of a three-phase line whose total constants are $Z = 150 \angle 85°$ ohms and $Y = 0.00080 \angle 90°$ mho to neutral, if the sending-end voltage is 165,000 between lines and the receiving-end power is 45,000 kw at 0.866 power factor, lagging current.

Prob. 14–5. In the line of problem 13–5, synchronous reactors are to be operated in parallel with the load at the receiving end to maintain a voltage of 160,000. If the reactor power factor is 0.03, what rating of reactors is required? The sending-end voltage is to remain 165,000.

Prob. 15–5. Prove that $E_S = I_S \dfrac{1}{Y \dfrac{\tanh \theta}{\theta}} - I_R \dfrac{1}{Y \dfrac{\sinh \theta}{\theta}}$.

Prob. 16–5. Prove that $E_R = I_S \dfrac{1}{Y \dfrac{\sinh \theta}{\theta}} - I_R \dfrac{1}{Y \dfrac{\tanh \theta}{\theta}}$.

Prob. 17–5. Derive the long-line current and voltage equations by the following method of successive approximations. First assume that I_R flows throughout the entire line, and determine the voltage distribution to correspond, letting receiver voltage be E_R. Corresponding to this voltage distribution, write an expression for the current distribution, and so on.

CHAPTER VI

CIRCLE DIAGRAMS

There are many ways in which the characteristics of transmission circuits may be shown graphically, and it is indeed fortunate that a considerable number of these graphs are circles. Since a circle can be drawn precisely if its radius and center are determined, it becomes possible to plot entire characteristics with a minimum of labor, and without the necessity of determining individually numerous points on each curve.

57. General Circuit Constants. When circle diagrams are to be prepared for a transmission circuit, they should usually be based on the entire circuit, rather than merely on the line itself. This ordinarily involves taking into account the transformer constants, and sometimes the line itself may consist of two or more portions having different constants. For some purposes, the generator parameters also need to be included.

No matter how complicated the transmission system may be, if all its elements are passive and linear, and it has only two terminals (in addition to ground or neutral), then the current and voltage at one end can be expressed as a simple linear function of the current and voltage at the other end, for balanced conditions, as follows:

$$E_S = AE_R + BI_R, \tag{249}$$

$$I_S = DI_R + CE_R; \tag{250}$$

and

$$E_R = DE_S - BI_S, \tag{251}$$

$$I_R = AI_S - CE_S. \tag{252}$$

In the foregoing equations, A, B, C and D are complex constants, called the general circuit constants of the system. A perfectly general way of working out their numerical values is to start at one end; for example, the receiving end, where the voltage and current are designated as E_R and I_R. Then, with all the parameters of the circuit known, it is possible to work back step by step to the sending end, adding successive drops to the voltage and adding successive shunt currents to the current, and applying the long-line formulas where necessary. The final expressions for sending-end voltage and current will be of the forms (249) and (250), and the general circuit constants are thus determined.

It can be seen by comparison with the long-line formulas that the values of the general circuit constants for a smooth line only, exclusive of terminal apparatus, are:

$$A = \cosh \theta, \tag{253}$$

$$B = Z \frac{\sinh \theta}{\theta}, \tag{254}$$

$$C = Y \frac{\sinh \theta}{\theta}, \tag{255}$$

$$D = \cosh \theta. \tag{256}$$

It is seen that A and D are equal, and this is true for all symmetrical circuits, that is, where the impedance is the same looking in from both ends.

The most common transmission circuit consists of a smooth line with a bank of transformers at each end. Let A_1, B_1, C_1 and D_1 represent the general circuit constants of the line itself, Z_{TS} the series impedance and Y_{TS} the no-load admittance of the sending-end transformer bank, per phase to neutral, and Z_{TR} and Y_{TR} the corresponding constants of the receiving-end transformers. Then, if we use the "cantilever" equivalent circuit for the transformers, the general circuit constants A, B, C and D for the entire circuit including the two terminal transformer banks are given by the following expressions (see figure 57):

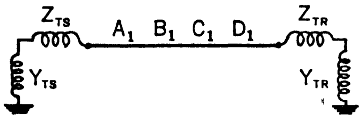

FIG. 57. Long line or any general linear network terminated by transformers.

$$A = A_1(1 + Z_{TR}Y_{TR}) + B_1Y_{TR} + C_1Z_{TS}(1 + Z_{TR}Y_{TR}) + D_1Z_{TS}Y_{TR}. \tag{257}$$

$$B = A_1Z_{TR} + B_1 + C_1Z_{TS}Z_{TR} + D_1Z_{TS}. \tag{258}$$

$$C = A_1Y_{TS}(1 + Z_{TR}Y_{TR}) + B_1Y_{TS}Y_{TR} + C_1(1 + Z_{TS}Y_{TS})(1 + Z_{TR}Y_{TR}) + D_1Y_{TR}(1 + Z_{TS}Y_{TS}). \tag{259}$$

$$D = A_1Z_{TR}Y_{TS} + B_1Y_{TS} + C_1Z_{TR}(1 + Z_{TS}Y_{TS}) + D_1(1 + Z_{TS}Y_{TS}). \tag{260}$$

If the line itself is symmetrical, making A_1 and D_1 equal, then terms containing those quantities can be combined. If the transformer impedances and admittances at the two ends are also equal, further combination is possible, and A becomes equal to D.

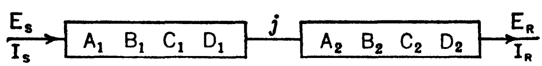

FIG. 58. Two general networks in series.

Two systems in series. The general circuit constants A, B, C and D for two general networks (see figure 58) in series are given by the expressions:

$$A = A_1A_2 + B_1C_2, \tag{261}$$
$$B = A_1B_2 + B_1D_2, \tag{262}$$
$$C = A_2C_1 + C_2D_1, \tag{263}$$
$$D = B_2C_1 + D_1D_2. \tag{264}$$

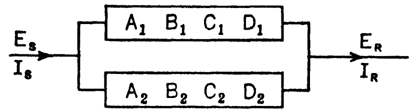

FIG. 59. Two general networks in parallel.

Two systems in parallel. The general circuit constants A, B, C and D for two general networks (see figure 59) in parallel are given by the expressions:

$$A = \frac{A_1B_2 + B_1A_2}{B_1 + B_2}, \tag{265}$$

$$B = \frac{B_1B_2}{B_1 + B_2}, \tag{266}$$

$$C = C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2}, \tag{267}$$

$$D = \frac{B_1D_2 + D_1B_2}{B_1 + B_2}. \tag{268}$$

Example. Prove the validity of expressions (261) and (262).

Solution. Referring to figure 58, the junction of the two networks is the sending end for the one on the right, and the current and voltage there, E_j and I_j, are:

$$E_j = A_2E_R + B_2I_R, \tag{269}$$
$$I_j = D_2I_R + C_2E_R. \tag{270}$$

These values E_j and I_j are at the receiving end of the left-hand network, and

$$\begin{aligned} E_S &= A_1E_j + B_1I_j \\ &= A_1(A_2E_R + B_2I_R) + B_1(D_2I_R + C_2E_R) \\ &= (A_1A_2 + B_1C_2)\,E_R + (A_1B_2 + B_1D_2)\,I_R. \end{aligned} \tag{271}$$

These final coefficients of E_R and I_R are the values of A and B as given in (261) and (262).

The equivalent T connection for the transformers is a slightly more accurate representation than the cantilever, but when there is a transformer bank at both ends, there is a partial cancellation of the small errors involved in the cantilevers. This occurs because at the sending end the no-load transformer current is assumed to flow through none of the series impedance, and at the receiving end it flows through all of it. Actually, it should flow through about half the series impedance at each end, representing the impedance of each primary winding. The total error involved is of the order of 0.1 per cent.

A relation which always exists among the four general circuit constants, and one which is of frequent value for checking, is that

$$AD - BC = 1. \tag{272}$$

58. Equivalent π and T. Just as it is possible to represent the characteristics of any two-terminal linear passive network by means of four general circuit constants, so also is it possible to reduce the network to a single π or T line which is the exact equivalent between terminals in the steady state and at rated frequency.

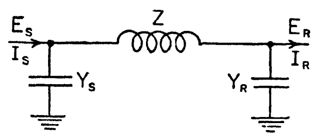

Fig. 60. Equivalent π of a general transmission circuit.

If the general circuit constants are known, the corresponding constants of the equivalent π (refer to figure 60) are given by the following equations:

$$Z = B, \tag{273}$$

$$Y_S = \frac{D - 1}{B}, \tag{274}$$

$$Y_R = \frac{A - 1}{B}. \tag{275}$$

If the constants of the π are known, the general circuit constants may be expressed in terms of them, as follows:

$$A = 1 + ZY_R, \tag{276}$$

$$B = Z, \tag{277}$$

$$C = Y_S + Y_R + ZY_SY_R, \tag{278}$$

$$D = 1 + ZY_S. \tag{279}$$

The equivalent π constants may be determined also by applying the equivalent Y-Δ and Δ-Y transformation formulas to the simplification of the original network.

In constructing models, the equivalent π method is almost indispensable, but for computation the equivalent circuit constants are more widely favored and in this book will be used as the basis for the circle diagrams.

59. Receiving-End Power Circle Diagram. We have from equation (249) that

$$E_S = AE_R + BI_R.$$

Let it be assumed that E_S and E_R are fixed in magnitude, and let E_R lie along the reference axis. The angle of E_S may vary, and I_R may vary in both magnitude and angle. These relations are shown graphically in figure 61. This operating condition is realized when the sending-end voltage is held constant by a regulator, controlling the generator field currents, and the receiving-end voltage is held constant by synchronous reactors whose field currents also are controlled by a regulator. The sending-end voltage, by the conditions imposed, is constrained to have its terminal point lie somewhere on the dotted circle, which is

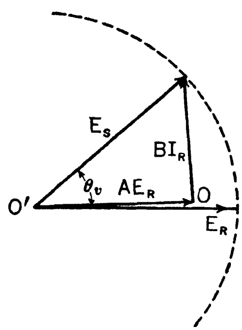

FIG. 61. Graphical representation of the equation $E_S = AE_R + BI_R$.

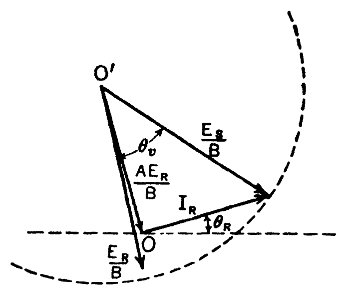

FIG. 62. Graphical representation of the equation $\frac{E_S}{B} = \frac{A}{B}E_R + I_R$.

drawn with center at O' and radius equal to the magnitude of E_S. Point O is fixed, and the vector BI_R must have its terminal point also on the dotted circle.

The physical significance of AE_R may be understood by noting that it is equal to the sending-end voltage when the line delivers no load, that is, when I_R and hence BI_R are zero.

If the scale is changed in figure 61 by dividing each vector by B, then the dotted circle becomes the locus of the terminal points of the vectors representing all possible receiving-end currents, the starting points of all the vectors being point O. Since B is a complex quantity in general, the division will shift the diagram through an angle, giving the relations shown in figure 62. The reference axis in figure 62 is the active component of receiving-end current. The angle θ_R shown in the figure is the phase angle of receiver current, shown leading. It is seen that the dotted circle now describes the locus of the terminal points of receiving-end currents which are possible under the conditions originally stated. What we have in this figure is a *load current circle diagram.* With reference to O as an origin and in-phase receiver current as an axis of reference, the circle center O' is seen to be located at $-\frac{A}{B}E_R$, the sign being negative owing to the fact that the sense of the vector $O'O$ has been reversed, and it is being measured from O to O'. It is more convenient to draw the relations of figure 62 in the manner illustrated in figure 63. Here the location of the center O' is given by the coordinates whose magnitudes are $-lE_R$, $+mE_R$ along the active and reactive current axes, respectively. To express these two new symbols l and m in terms of the general circuit constants, let

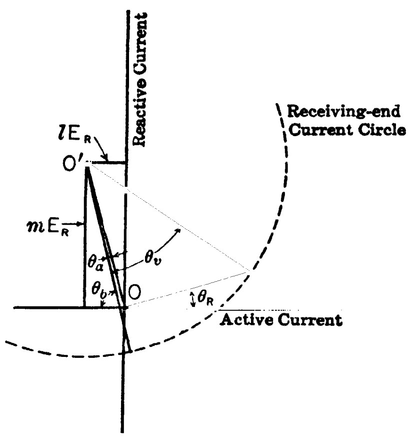

FIG. 63. Receiving-end current circle diagram.

$$A = a\underline{|\theta_a} = a(\cos\theta_a + j\sin\theta_a), \tag{280}$$

$$B = b\underline{|\theta_b} = b(\cos\theta_b + j\sin\theta_b), \tag{281}$$

$$C = c\underline{|\theta_c} = c(\cos\theta_c + j\sin\theta_c), \tag{282}$$

$$D = d\underline{|\theta_d} = d(\cos\theta_d + j\sin\theta_d). \tag{283}$$

Then $-\frac{A}{B}E_R = -\frac{a}{b}\underline{|\theta_a - \theta_b}\,E_R$, and the coordinates of O' are

$$lE_R = -\frac{a}{b}E_R\cos(\theta_b - \theta_a), \tag{284}$$

$$mE_R = \frac{a}{b}E_R\sin(\theta_b - \theta_a). \tag{284a}$$

The radius of the load current circle is

$$\text{Radius} = \frac{E_S}{b}. \tag{285}$$

It may be noticed in figure 61 that, if the vector E_S swings around to the reference axis, we have the condition where there is no phase difference between the voltages at the two ends of the line. The angle between them in the diagram is marked θ_v, and this angle is shown also in figures 62 and 63.

The current circle diagram is convertible to the power circle diagram by simple multiplication of all the scales and vectors by factor E_R, or preferably 0.003 E_R in order to convert to units of kilovolt-amperes and kilowatts for all three phases. On this basis, the receiving-end power circle has its center at

$$-0.003\ E_R{}^2 \frac{a}{b} \cos(\theta_b - \theta_a) \text{ kw}, \tag{286}$$

$$+0.003\ E_R{}^2 \frac{a}{b} \sin(\theta_b - \theta_a) \text{ rkva}, \tag{287}$$

and radius

$$0.003 \frac{E_S E_R}{b} \text{ kva}. \tag{288}$$

60. Sending-End Power Circle Diagrams. We have from equation (251) the relation that

$$E_R = DE_S - BI_S.$$

Assuming that E_S and E_R are again fixed in magnitude, and that E_S now lies along the reference axis, we may express the relation graphically by means of the vector diagram of figure 64. Physically, the vector DE_S represents the value which E_R must take on when there is no sending-end current, making $BI_S = 0$.

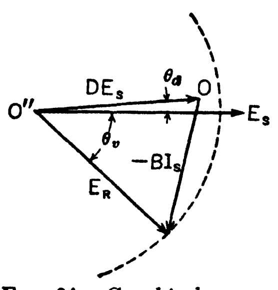

FIG. 64. Graphical representation of the equation $E_R = DE_S - BI_S$.

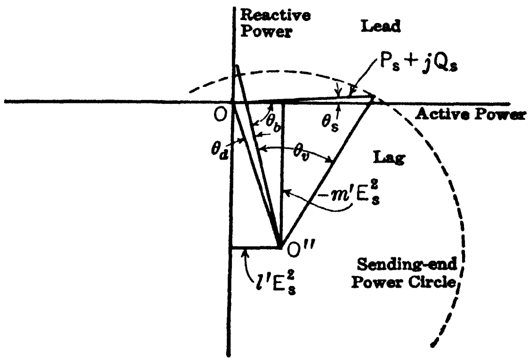

FIG. 65. Sending-end power circle diagram.

By the same process of scale changing as used before in developing the receiving-end circle, the diagram of figure 64 may be converted to the sending-end power circle diagram of figure 65. The location of the center of the sending-end power circle has as coordinates

$$+0.003\ E_S{}^2 \frac{d}{b} \cos(\theta_b - \theta_d) \text{ kw}, \tag{289}$$

$$-0.003\ E_S{}^2 \frac{d}{b} \sin(\theta_b - \theta_d) \text{ rkva}, \tag{290}$$

and its radius is

$$0.003 \frac{E_S E_R}{b} \text{ kva.} \tag{291}$$

The convention is used that inductive or lagging reactive power is negative, and leading reactive power positive.

61. Combined Sending and Receiving Power Circle Diagram. It is sometimes convenient to draw both sending and receiving power circle diagrams on a single chart. A common set of axes, representing active and reactive power, is used. There is not really a common voltage axis, because the active component of sending power is supplied by the

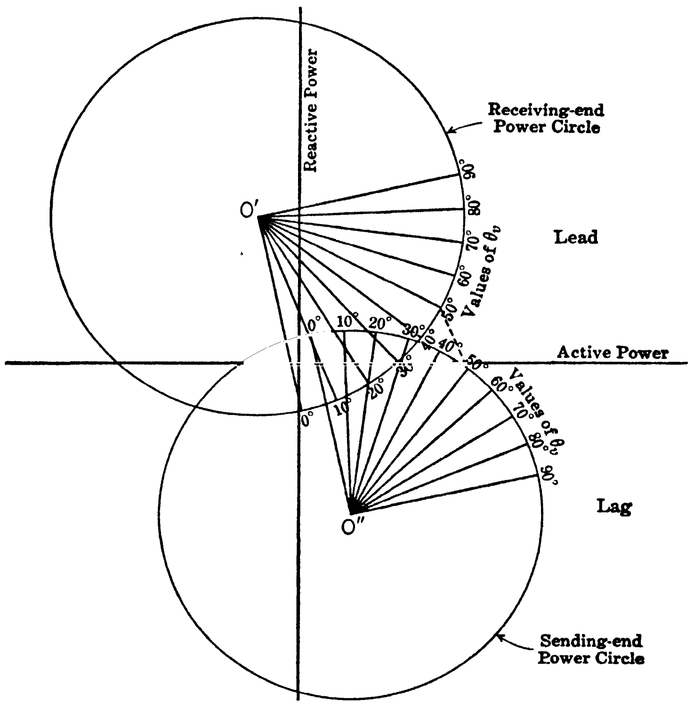

FIG. 66. Receiving-end and sending-end power circles, with voltage angle scales.

current in phase with the sending-end voltage, whereas the active component of receiver power is supplied by the current in phase with the receiver voltage. These two voltages are not, in general, in phase with each other. We may, however, mark on each circle the operating point representing the condition of operation when the terminal voltages are in phase. The operating points on the two circles corresponding to a single operating condition can be determined by laying off the same angle θ_v from the reference marks on the two circles. Referring to figure 66, in which both receiving-end and sending-end circles are shown, voltage angle scales have been drawn on both circles. A dotted line has been drawn to connect the two operating points which correspond to a phase difference of 50 degrees in the terminal voltages. This line represents the vector power lost in the transmission circuit. The difference in active power is the amount of the transmission losses; and the vertical component of the dotted line, representing reactive power loss, indicates that the line (including transformers, if they have been included in determining the general circuit constants) takes a net lagging reactive power. At lighter loads, the line is found to take a net leading reactive power.

The whole chart, it must be remembered, is premised on a constant magnitude of voltage at each end.

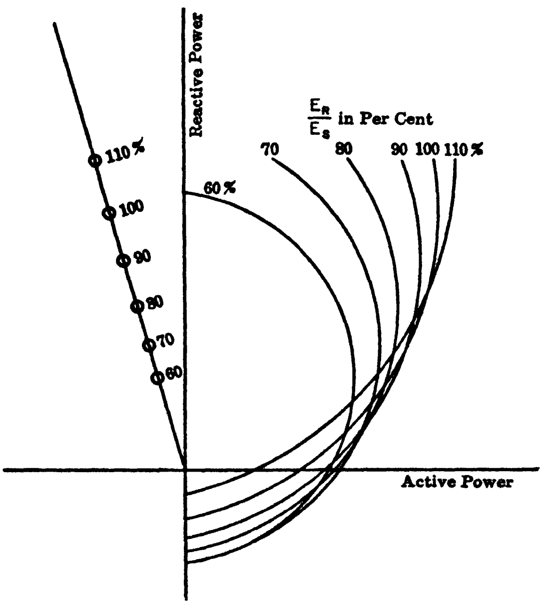

FIG. 67. Receiving-end power chart for a fixed value of E_S and several values of E_R.

62. Variable Terminal Voltages. The power circle diagrams have been developed on the basis of constant terminal voltages E_S and E_R, but their use is by no means restricted to this case. If the sending-end voltage is actually fixed, but the receiving-end voltage a variable dependent on the load, then a chart such as shown in figure 67 may be used.

A circle is drawn for each of a number of different receiving-end voltages. These circles serve to map out the working area of the chart so that, for any given vector receiving-end power, it is possible to read directly, by visual interpolation between the circles, the magnitude of the receiving-end voltage.

As a matter of fact, two receiving-end voltages are possible for a fixed size of E_S and a given vector power at the receiving end. This point was brought out in the first example of article 52 in the preceding chapter. In the graphical solution, the two possible voltages are given by the readings on the large and small power circles which overlap. The low reading is not ordinarily of practical importance.

The centers of the different circles in figure 67 all lie along a straight line, and their distances from the origin are in proportion to the squares of the respective receiving-end voltages. The radii of the circles are in direct proportion to the first power of the receiving-end voltages.

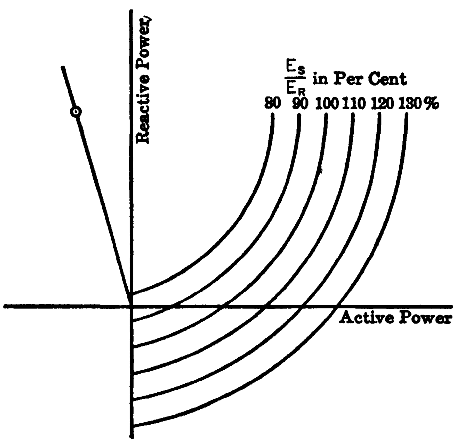

FIG. 68. Receiving-end power chart for a fixed value of E_R and several values of E_S.

A somewhat similar chart may be drawn to cover the condition of a fixed receiving-end voltage and various different values of sending-end voltage. A receiving-end power chart of this type is shown in figure 68.

For some purposes, including the study of transient conditions involving swings in phase angle and sizes of terminal voltages, it is desirable to construct charts from which readings may be made for changes in both E_S and E_R. Of course a whole family of charts of the type of either figure 67 or 68 could be constructed, but it is also possible, by a modification of the scales, to accomplish the purpose with but a single chart. It will be recalled that the center of the receiving-end power circle is located at

$$-0.003\ E_R^2 \frac{a}{b} \cos\ (\theta_b - \theta_a)\ \text{kw},$$

$$+0.003\ E_R^2 \frac{a}{b} \sin\ (\theta_b - \theta_a)\ \text{rkva},$$

and that the radius is equal to

$$0.003 \frac{E_S E_R}{b} \text{ kva.}$$

If the scales are changed by dividing by a factor E_R^2, the modified receiving-end power circle will have its center located at

$$-0.003 \frac{a}{b} \cos (\theta_b - \theta_a), \tag{292}$$

$$+0.003 \frac{a}{b} \sin (\theta_b - \theta_a), \tag{293}$$

and its radius will be

$$\frac{0.003}{b} \frac{E_S}{E_R} \tag{294}$$

The axes of coordinates, instead of being measured in units of kilowatts and reactive kilovolt-amperes, are converted to units of kw/E_R^2 and rkva/E_R^2. It is observed that the center is fixed, independent of either terminal voltage, and that the circle radius depends on the ratio of voltages rather than the actual magnitude of either. In reading the chart, the value of receiving-end power is determined by multiplying the coordinate readings by the appropriate values of E_R^2.

Charts for sending-end power, analogous to those described for the receiving end, may of course be constructed.

63. Loss Circles. If curves of constant loss are plotted on a chart such as that of figure 68, these loss curves will be concentric circles. To prove this, and to determine the center and radii of the circles, let us set up the expression for the loss. The following notation will be used:

$E_R = E_R + j\,0$ volts, taken as axis.
$I_R = I_{R1} + jI_{R2}$ amp.
$E_S = E_{S1} + jE_{S2}$ volts.
$I_S = I_{S1} + jI_{S2}$ amp.
$A = a_1 + ja_2$.
$B = b_1 + jb_2$.
$C = c_1 + jc_2$.
$D = d_1 + jd_2$.
P_R = watts receiving-end active power per phase.
P_S = watts sending-end active power per phase.
$P_L = P_S - P_R$ watts loss active power per phase.

Starting with the general equations, we can proceed as follows:

$$E_S = AE_R + BI_R$$
$$= (a_1E_R + b_1I_{R1} - b_2I_{R2}) + j(a_2E_R + b_1I_{R2} + b_2I_{R1}). \tag{295}$$

$$
\begin{aligned}
I_S &= DI_R + CE_R \\
&= (d_1 I_{R1} + c_1 E_R - d_2 I_{R2}) + j(d_1 I_{R2} + d_2 I_{R1} + c_2 E_R). && (296) \\
P_R &= E_R I_{R1}. && (297) \\
P_S &= E_{S1} I_{S1} + E_{S2} I_{S2} \\
&= E_R^2\,(a_1c_1 + a_2c_2) + E_R I_{R1}(a_1d_1 + a_2d_2 + b_1c_1 + b_2c_2) \\
&\quad + E_R I_{R2}(a_2d_1 - a_1d_2 + b_1c_2 - b_2c_1) + I_{R1}^2(b_1d_1 + b_2d_2) \\
&\quad + I_{R2}^2(b_1d_1 + b_2d_2). && (298)
\end{aligned}
$$

From equation (272) we have the relation that $AD - BC = 1$. If this expression is expanded in terms of its real and imaginary components, the following two relations appear:

$$a_1d_1 - a_2d_2 - b_1c_1 + b_2c_2 = 1, \qquad (299)$$

$$a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1 = 0. \qquad (300)$$

We can then write:

$$
\begin{aligned}
P_L &= P_S - P_R \\
&= E_R^2(a_1c_1 + a_2c_2) + E_R I_{R1}(2\,a_2d_2 + 2\,b_1c_1) \\
&\quad + E_R I_{R2}(-2\,a_1d_2 + 2\,b_1c_2) + I_{R1}^2(b_1d_1 + b_2d_2) \\
&\quad + I_{R2}^2(b_1d_1 + b_2d_2). && (301)
\end{aligned}
$$

Rearranging (301) to put it in the standard form for the equation of a circle, we have:

$$
\left(I_{R1} + E_R\frac{a_2d_2 + b_1c_1}{b_1d_1 + b_2d_2}\right)^2 + \left(I_{R2} - E_R\frac{a_1d_2 - b_1c_2}{b_1d_1 + b_2d_2}\right)^2
= \frac{\left(P_L - E_R^2\dfrac{b_1^2(a_1 - 1) + a_2b_2b_1}{b_1^2 + b_2^2}\right)(b_1d_1 + b_2d_2) - E_R^2\left[\dfrac{b_1^2(d_1 - 1) + d_2b_2b_1}{b_1^2 + b_2^2}\right]}{(b_1d_1 + b_2d_2)^2} \qquad (302)
$$

On the receiving-end current diagram, this indicates that the line which is the locus of a loss P_L watts per phase is a circle whose center is located at

$$-E_R\frac{a_2d_2 + b_1c_1}{b_1d_1 + b_2d_2} \quad \text{active amp,} \qquad (303)$$

$$+E_R\frac{a_1d_2 - b_1c_2}{b_1d_1 + b_2d_2} \quad \text{reactive amp,} \qquad (304)$$

and whose radius is equal to

$$
\frac{\sqrt{\left\{\left[P_L - E_R^2\dfrac{b_1^2(a_1 - 1) + b_1a_2b_2}{b_1^2 + b_2^2}\right](b_1d_1 + b_2d_2) - E_R^2\left[\dfrac{b_1^2(d_1 - 1) + d_2b_2b_1}{b_1^2 + b_2^2}\right]\right\}}}{b_1d_1 + b_2d_2} \text{ amp.} \qquad (305)
$$

It will be noticed that the center of all loss circles for a given value of E_R is fixed, but the radius is a function of the loss. The minimum loss for a given active receiving-end power is attained when the reactive component of the receiving-end current is equal to the reactive component of the center of the loss circles.

Since the expression for the radii of the loss circles is rather complicated, computation of loss may most frequently be done to best advantage by direct calculation of P_S and P_R, even though this involves taking a difference of two numbers nearly equal to each other. If a large number of loss calculations is to be made, it may be best to draw the loss circles.

On coordinate axes whose units are receiving-end kilowatts and reactive kilowatt-amperes for all three phases, the loss circle center is at

$$-0.003\ E_R^2 \frac{a_2d_2 + b_1c_1}{b_1d_1 + b_2d_2} \text{kw}, \tag{306}$$

$$+0.003\ E_R^2 \frac{a_1d_2 - b_1c_2}{b_1d_1 + b_2d_2} \text{rkva}. \tag{307}$$

The radius for a loss of P_L kilowatts in all three phases is equal to

$$0.003\ E_R \sqrt{\frac{\left\{\left[\frac{1000\ P_L}{3} - E_R^2 \frac{b_1(a_1 - 1) + a_2b_2}{b_1^2 + b_2^2}\right](b_1d_1 + b_2d_2) - E_R^2\left[\frac{b_1^2(d_1 - 1) + d_2b_2b_1}{b_1^2 + b_2^2}\right]\right\}}{b_1d_1 + b_2d_2}}\ \text{kva}. \tag{308}$$

64. Straight-Line Loss Diagrams. If the transmission system is operating with constant voltage at both ends, it is possible to augment the power circle diagram by the addition of a straight line, from which perpendiculars (to the line) may be extended to the power circle and used as measures of the loss. Referring to figures 63 and 65, we may write

$$P_S = l'E_S^2 - \frac{E_RE_S}{b}(\cos\theta_b \cos\theta_v - \sin\theta_b \sin\theta_v); \tag{309}$$

$$P_R = -lE_R^2 + \frac{E_RE_S}{b}(\cos\theta_b \cos\theta_v + \sin\theta_b \sin\theta_v). \tag{310}$$

$$P_L = P_S - P_R = lE_R^2 + l'E_S^2 - 2\frac{E_RE_S}{b}\cos\theta_b \cos\theta_v$$

$$\text{watts per phase.} \tag{311}$$

For fixed values of E_R and E_S, the only variable on the right in (311) is θ_v, the phase angle between the voltages E_R and E_S. The loss is thus expressed as the difference between a constant term $lE_R^2 + l'E_S^2$ and the component of a vector of length $2\,\frac{E_R E_S}{b}\cos\theta_b$ projected on an axis with which it makes an angle θ_v. This relation is illustrated in figure 69. The vertical line is the axis of $\theta_v = 0$.

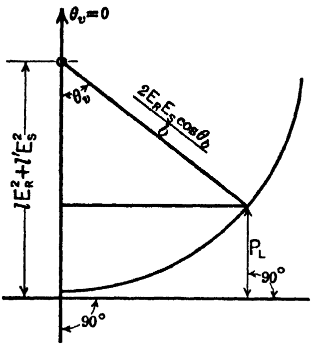

FIG. 69. Graphical construction for determining loss P_L as a function of θ_v, from (311).

If a loss scale is selected such as to make the radius of the loss circle $2\,\frac{E_R E_S \cos\theta_b}{b}$ equal to the radius of the power circles $\frac{E_R E_S}{b}$, then the two may be superimposed on a single chart, either for the sending-end or receiving-end conditions or both. Obviously, to attain this result the loss scale must be made greater than the main power scale by a factor $2\cos\theta_b$, and then the power circle itself may be used as the loss circle. This usually means that the loss scale will be three or four times larger

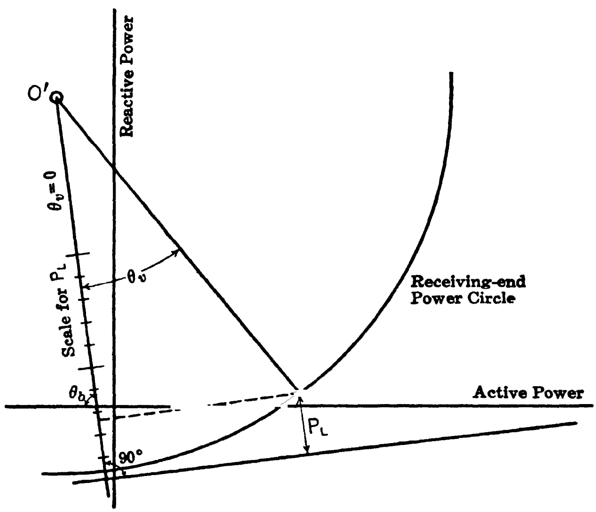

FIG. 70. Loss chart superimposed on receiving-end power chart.

than the main power scale, which is a favorable condition, since the loss to be read is much smaller than the total power. The superimposed charts for receiving-end conditions are shown in figure 70. In just the same way the loss chart can be superimposed on the sending-end power chart.

65. Efficiency Circles. Curves of constant efficiency plotted on a receiving-end power scale, for constant receiving-end voltage, are circles, but are not concentric. Such curves are not of much practical use, since the effect of the losses on design and operation is more readily determined from the losses themselves than from the efficiency.

66. Résumé of Circle-Diagram Formulas. Based on coordinate scales of kilowatts and reactive kilowatt-amperes for all three phases, the following are the principal formulas for the construction of power and loss diagrams.

1. **Receiving-end power circles.** The center is located at

$$-0.003\ E_R^2 \frac{a}{b} \cos\ (\theta_b - \theta_a) \text{ kw},$$

$$+0.003\ E_R^2 \frac{a}{b} \sin\ (\theta_b - \theta_a) \text{ rkva}.$$

The radius is

$$0.003 \frac{E_S E_R}{b} \text{ kva}.$$

The voltages are to neutral, and powers for all three phases.

2. **Sending-end power circles.** The center is located at

$$+0.003\ E_S^2 \frac{d}{b} \cos\ (\theta_b - \theta_d) \text{ kw},$$

$$-0.003\ E_S^2 \frac{d}{b} \sin\ (\theta_b - \theta_d) \text{ rkva}.$$

The radius is

$$0.003 \frac{E_S E_R}{b} \text{ kva}.$$

3. **Loss circles.** The center of the loss circles on the receiving-end power chart is located at

$$-0.003\ E_R^2 \frac{a_2 d_2 + b_1 c_1}{b_1 d_1 + b_2 d_2} \text{kw},$$

$$+0.003\ E_R^2 \frac{a_1 d_2 - b_1 c_2}{b_1 d_1 + b_2 d_2} \text{rkva}.$$

The radius for P_L total kilowatt loss is

$$0.003\ E_R \sqrt{\frac{\left\{\left[\frac{1000\ P_L}{3} - E_R^2 \frac{b_1(a_1 - 1) + a_2 b_2}{b^2}\right](b_1 d_1 + b_2 d_2) - E_R^2 \frac{b_1^2(d_1 - 1) + d_2 b_2 b_1}{b^2}\right\}}{b_1 d_1 + b_2 d_2}} \text{ kva}.$$

4. **Loss line.** The loss line may be added to a receiving-end power circle diagram by drawing the line of zero θ_v through the center O' of the power circle and erecting a perpendicular on this line at a point distant from O' by

$$\frac{0.003\left[E_R^2\frac{a}{b}\cos(\theta_b-\theta_a)+E_S^2\frac{d}{b}\cos(\theta_b-\theta_d)\right]}{2\cos\theta_b}\text{ kva}$$

measured to the scale of the main axes. The loss is measured by erecting a perpendicular to the loss line to intersect the power circle at the operating point. The length of this line between the circle and the loss line is the value of loss to a scale 2 cos θ_b greater than the coordinate scales.

The loss line may be added to a sending-end power circle diagram in the same way.

67. Numerical Examples. Construction of Charts. A 200-mile 60-cps single-circuit three-phase line has three 715,500-cir-mil 54 + 7 strand A.C.S.R. spaced 25 ft apart in a horizontal plane whose average height above ground is 40 ft. At each end of the line is a bank of two-winding transformers having the following constants per phase: $Z_T = 34.5\,\underline{|86.4^\circ}$ ohms and $Y_T = 0.93 \times 10^{-4}\,\underline{|-83.5^\circ}$ mho to neutral. The constants are referred to the high-tension side.

1. Determine the constants A, B, C and D for the entire circuit.
2. Prepare a receiving-end power diagram for a sending-end voltage of 220,000 and a load-end voltage of 200,000 between lines.
3. Draw a set of loss circles on a receiving-end power chart for a load voltage of 200,000 between lines.
4. Draw the loss diagram on the receiving-end power diagram.

Solution 1. General circuit constants. The value of self gmd from table V for 715,500-cir-mil A.C.S.R. is 0.414 in. The mutual gmd is $25\sqrt[3]{2}$ ft or 378 in. The ratio D_m/D_s is 914, and from table VII the reactance per mile at 60 cps is 0.827 ohm. The reactance of the 200 miles is 165.4 ohms per phase. The resistance is 0.133 × 200 = 26.6 ohms at 25 C. The capacitive susceptance per mile for this line has been worked out in Chapter IV, and is equal to 5.21×10^{-6} mho per mile, which corresponds to 1.042×10^{-3} mho for 200 miles. We have then

$$Z = 26.6 + j\,165.4 = 167.6\,\underline{|80.86^\circ}\text{ ohms.}$$

$$Y = 0 + j\,1.042 \times 10^{-3} = 0.001042\,\underline{|90^\circ}\text{ mho.}$$

$$ZY = 0.1748\,\underline{|170.86^\circ}.$$

$$\cosh\theta = 0.915\,\underline{|0.86^\circ}.$$

$$\frac{\sinh\theta}{\theta} = 0.9715\,\underline{|0.28^\circ}.$$

$$A_1 = \cosh\theta = 0.915\,\underline{|0.86^\circ}.$$

$$B_1 = Z\frac{\sinh\theta}{\theta} = 162.8\,\underline{|81.14^\circ}\text{ ohms.}$$

$$C_1 = Y\frac{\sinh\theta}{\theta} = 0.001012\,\underline{|90.28^\circ}\text{ mho.}$$

$$D_1 = \cosh\theta = 0.915\,\underline{|0.86^\circ}.$$

The constants A_1, B_1, C_1 and D_1 are for the line alone. The general circuit constants A, B, C and D for the entire circuit, line and transformers, are found from (257), (258), (259) and (260). For the symmetrical arrangement which occurs in this problem, some simplification is possible. We have $Z_{TR} = Z_{TS} = Z_T$, and $Y_{TR} = Y_{TS} = Y_T$. A_1 and D_1 are equal, and we shall also have A and D equal. The following simplified forms may be used in the calculation.

$$B = B_1 + 2\,A_1Z_T + C_1Z_T^2; \tag{312}$$

$$A = D = A_1 + BY_T + C_1Z_T; \tag{313}$$

$$C = C_1 + 2\,AY_T - BY_T^2. \tag{314}$$

Substituting the numerical values, we find

$$\begin{aligned} B &= 162.8\,\underline{|81.14^\circ} + 2 \times 0.915\,\underline{|0.86^\circ} \times 34.5\,\underline{|86.4^\circ} \\ &\qquad + 0.001012\,\underline{|90.28^\circ} \times 34.5^2\,\underline{|2 \times 86.4} \\ &= 162.8\,\underline{|81.14^\circ} + 63.1\,\underline{|87.26^\circ} + 1.2\,\underline{|263.08^\circ} \\ &= 28.0 + j\,222.7 = 224.5\,\underline{|82.84^\circ}\text{ ohms.} \end{aligned}$$

$$\begin{aligned} A = D &= 0.915\,\underline{|0.86^\circ} + 224.5\,\underline{|82.84^\circ} \times 0.93 \times 10^{-4}\,\underline{|-83.5^\circ} \\ &\qquad + 0.001012\,\underline{|90.28^\circ} \times 34.5\,\underline{|86.4^\circ} \\ &= 0.915\,\underline{|0.86^\circ} + 0.0209\,\underline{|-0.66^\circ} + 0.0350\,\underline{|176.68^\circ} \\ &= 0.901 + j\,0.01553 = 0.901\,\underline{|0.99^\circ}. \end{aligned}$$

$$\begin{aligned} C &= 0.001012\,\underline{|90.28^\circ} + 2 \times 0.901\,\underline{|0.99^\circ} \times 0.93 \times 10^{-4}\,\underline{|-83.5^\circ} \\ &\qquad - 224.5\,\underline{|82.84^\circ} \times 0.93^2 \times 10^{-8}\,\underline{|-2 \times 83.5^\circ} \\ &= 0.001012\,\underline{|90.28^\circ} + 0.0001676\,\underline{|-82.51^\circ} + 0.0000019\,\underline{|95.84^\circ} \\ &= 0.0000167 + j\,0.000848 = 0.000848\,\underline{|88.87^\circ}\text{ mho.} \end{aligned}$$

Check: $AD - BC = 1 + j\,0.$

Solution 2. Receiving-end power circle. The center of the receiving-end power circle for $E_S = 220{,}000/\sqrt{3} = 127{,}000$, and $E_R =$

$200{,}000/\sqrt{3} = 115{,}500$ volts, has as its active component

$$-0.003\ E_R{}^2 \frac{a}{b} \cos\ (\theta_b - \theta_a)$$
$$= -4 \times 10^7 \frac{0.901}{224.5} \cos\ (82.84° - 0.99°)$$
$$= -22{,}700 \text{ kw.}$$

The reactive component is

$$+0.003\ E_R{}^2 \frac{a}{b} \sin\ (\theta_b - \theta_a).$$
$$= +158{,}800 \text{ rkva.}$$

The radius is

$$0.003 \frac{E_S E_R}{b}$$
$$= \frac{4.4 \times 10^7}{224.5}$$
$$= 196{,}000 \text{ kva.}$$

The receiving-end power circle diagram is shown in figure 71.

Solution 3. Loss circles. For a load voltage of 200,000 between lines ($E_R = 115{,}500$ volts) the active component of the loss circle center is

$$-0.003\ E_R{}^2 \frac{a_2 d_2 + b_1 c_1}{b_1 d_1 + b_2 d_2}$$
$$= -4 \times 10^7 \frac{0.01553^2 + 28.0 \times 0.0000167}{28.0 \times 0.901 + 222.7 \times 0.01553}$$
$$= -975 \text{ kw.}$$

The reactive component is

$$+0.003\ E_R{}^2 \frac{a_1 d_2 - b_1 c_2}{b_1 d_1 + b_2 d_2}$$
$$= +4 \times 10^7 \frac{0.901 \times 0.01553 - 28.0 \times 0.000848}{28.0 \times 0.901 + 222.7 \times 0.01553}$$
$$= -13{,}600 \text{ rkva.}$$

The expression for the radius of the loss circle, upon substitution of the numerical values, reduces to

$$1180\sqrt{P_L - 1340} \text{ kva.}$$

If P_L is less than 1340 kw, the formula indicates an imaginary value of the radius. This means that there is no possible value of load which can result in a loss less than 1340 kw, if the load voltage is maintained at 200,000 between lines. The radii for different amounts of loss are given in table XIX. The loss circles have been drawn in figure 71.

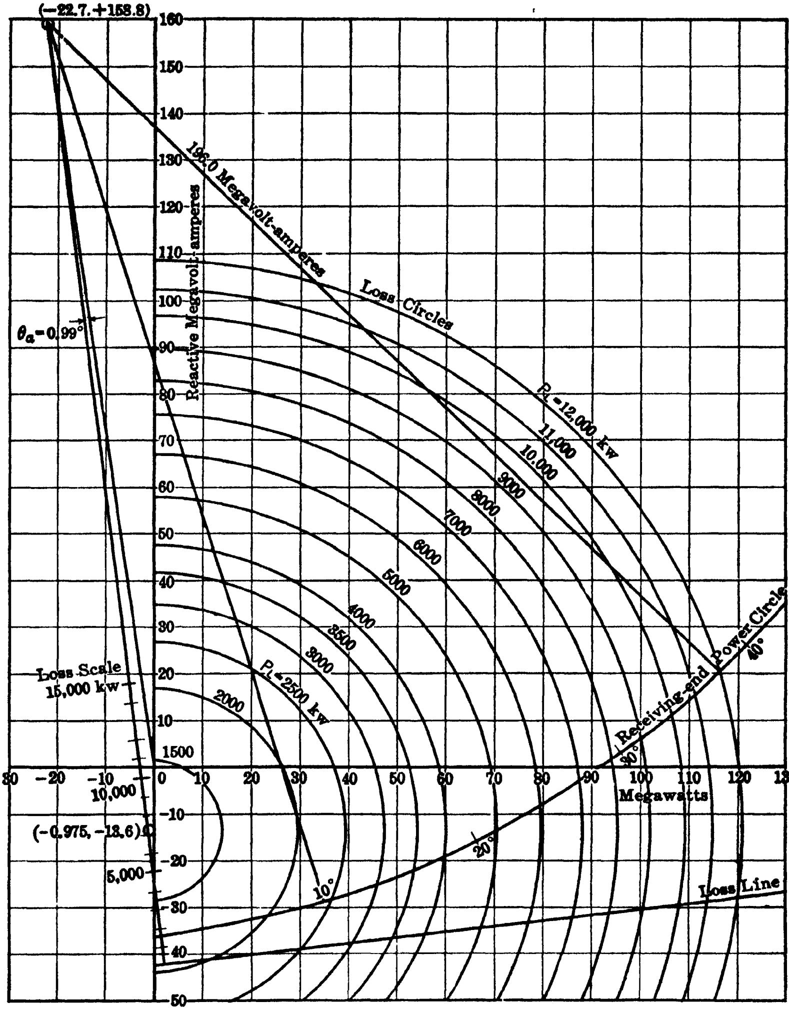

FIG. 71. Receiving-end power circle for $E_S = 220{,}000/\sqrt{3}$ and $E_R = 200{,}000/\sqrt{3}$ volts; loss line and loss circles.

TABLE XIX

Total Loss (kw)	Radius of Loss Circle (megavolt-amperes)	Total Loss (kw)	Radius of Loss Circle (megavolt-amperes)
1340	0	6000	80.6
1500	14.9	7000	88.8
2000	30.3	8000	96.3
2500	40.2	9000	103.2
3000	48.1	10,000	109.9
3500	54.9	11,000	116.0
4000	61.0	12,000	122.1
5000	71.5	13,000	127.8

It is to be observed that these circles do not apply only for a constant sending-end voltage, but for any steady condition as long as E_R remains fixed.

Solution 4. Loss line. The distance from the center of the receiving-end circle to the loss line is

$$\frac{0.003\left[E_R{}^2\frac{a}{b}\cos(\theta_b-\theta_a)+E_S{}^2\frac{d}{b}\cos(\theta_b-\theta_d)\right]}{2\cos\theta_b}$$

$$=\frac{8.84\times 10^7\dfrac{0.901}{224.5}\cos 81.85^\circ}{2\cos 82.84^\circ}$$

$$= 201{,}800 \text{ kva.}$$

The line of $\theta_v = 0$ is drawn making an angle $\theta_a = 0.99$ degree from the line joining the center of the receiving-end power circle with the origin. The distance 201,800 is measured down the line of $\theta_v = 0$, and the loss line is drawn perpendicular at this point. The loss scale is $2 \cos \theta_b = 0.249$ times the main power scale. The construction is shown in figure 71.

The two methods of indicating the loss, by circles and by loss line, may be checked against each other on the chart. The loss line shown, however, applies only for $E_S = 220{,}000/\sqrt{3}$, whereas the loss circles are not so restricted.

68. Numerical Examples. Graphical Solutions from Charts. The chart of figure 71 will be used as the basis for some illustrative examples.

Example 1. The load exclusive of synchronous reactors amounts to $90{,}000 - j\,40{,}000$ kva. What load will be taken by the synchronous reactors? The reactors operate at 2.0 per cent power factor.

Solution. Locate the load point at $90 - j\,40$ megawatts and draw a line making an angle $\cos^{-1} 0.02$ to intersect the power circle. The length of the line is 39,000 kva, which must be the reactor load.

Example 2. The load varies from zero to $100 - j\,50$ megawatts, always at the same power factor. If the reactors which are to be used can be loaded to their full rating leading, and 60 per cent of rating lagging, what is the minimum rating which will serve to maintain constant receiver voltage of 200,000 between lines? The sending-end voltage is to be kept constant, but may be given any desired value. Reactor power factor is 2.0 per cent.

Solution. A method of successive approximations will be used. With the power circle in figure 71, for 220,000 volts at the sending end, there would be required 56,000 reactor kva leading at the maximum load and 36,000 kva lagging at zero load. To handle 36,000 kva lagging would require a reactor rating of 36,000/0.60 = 60,000 kva.

A smaller power circle must be tried. If the radius is made less by 1500 kva, then the lagging reactor kva required at no load is reduced to 34,500, indicating a rating of 34,500/0.60 = 57,500 kva. The leading kva required at full load is the same amount. This is the minimum, and corresponds to a sending-end voltage of 218,300 between lines.

Example. The line of figure 71 supplies a continuous load of $60 - j\,50$ megavolt-amperes at 200,000 volts to its load. Neglecting any consideration of voltage control (since the load is continuous), compute how much if any reactor capacity should be used in parallel with the load for the purpose of reducing line losses. Total cost of installing reactors is $10 per kva; power factor of reactors is 2.0 per cent; value of lost power is 0.6 cent per kwhr; assume 12 per cent as annual fixed and operating costs of reactors, not including power losses.

Solution. A comparison of the annual cost of operation with different amounts of reactor capacity will be made. Costs that are independent of reactor rating will not be considered.

Reactor kva	Total kva	Kw Line Loss	Total Loss	Fixed Charge	Annual Value of Lost Power	Total Variable Cost
0	$60,000 - j\,50,000$	5000	5000	0	$263,000	$263,000
10,000	$60,200 - j\,40,000$	4500	4700	$12,000	247,000	259,000
20,000	$60,400 - j\,30,000$	4200	4600	24,000	242,000	266,000
30,000	$60,600 - j\,20,000$	4100	4700	36,000	247,000	283,000
40,000	$60,800 - j\,10,000$	4100	4900	48,000	258,000	306,000

The minimum annual cost is attained by using about 10,000 kva of reactor capacity for power-factor correction.

If the load varies during the day, as it usually does, the power circle which will result in minimum annual cost may be determined in a somewhat similar way. The variable load may be represented by a number of fixed loads, each assumed to persist over part of the day. Lost energy computed for each load and for an assumed voltage condition may be totaled, and annual costs determined. The computation needs to be repeated for different voltages and the condition corresponding to minimum annual cost thus selected.

Example. What is the power limit imposed by the line itself at the voltages specified?

Solution. The radius of the power circle is 196,000 kva, and its center has an active component of $-22,700$ kw. The maximum abscissa of the circle is thus $196,000 - 22,700 = 173,300$ kw, which is the limit of the line itself. The actual limit may be considerably smaller, owing to characteristics of terminal apparatus (generators and reactors) and the economic impracticability of providing enough reactor capacity to bring the operating point so high on the circle.

69. Power Diagrams with Compounding. It is sometimes desirable to provide means for increasing voltage with load either at the sending end alone, or at both ends of the line. Circle diagrams have been worked out for these cases, and the derivations may be found in some of the references.

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PROBLEMS ON CHAPTER VI

Prob. 1–6. Prepare a receiving-end power chart for the line described in article 67, for a sending-end voltage of 220,000 between lines, and draw power circles for receiving-end voltages between lines for each 10,000 volts from zero to 220,000. Use a scale of 10,000 kva to the inch.

Prob. 2–6. From the chart of problem 1–6, draw curves with receiving-end power as abscissas, and receiving-end voltage as ordinates, for each of the following power factors: unity; 0.90 lead and lag; 0.80 lead and lag.

Prob. 3–6. Plot a receiving-end power chart for a receiving-end voltage of 200,000 and sending-end voltages of 10,000, 20,000, 30,000, etc., up to 250,000 volts between lines. Use a scale of 10,000 kva to the inch.

Prob. 4–6. Draw the loss circles on the power chart of problem 3–6.

Prob. 5–6. Assuming a receiving-end voltage of 200,000 and using the chart of problems 3–6 and 4–6, determine the most economical value of generator voltage and rating of synchronous reactors at the receiving end for the following conditions. Customers' load is $100{,}000 - j\,50{,}000$ kva for 8 hours per day; $60{,}000 - j\,30{,}000$ for 10 hours per day, and $30{,}000 - j\,20{,}000$ for 6 hours per day, 365 days per year. Synchronous reactors cost $8 per kva complete; fixed charges are 15 per cent per year. Reactor power factor is 2.0 per cent; value of lost energy is 0.70 cent per kwhr. Line is to be operated with constant voltage at both ends.

Prob. 6–6. Draw the sending-end power chart for the line of article 67, for a sending-end voltage of 220,000 and receiving-end voltage of 200,000, between lines. Use a scale of 10,000 kva to the inch.

Prob. 7–6. Draw the loss line and loss scale on the chart of problem 6–6.

CHAPTER VII

TRAVELING WAVES

70. General Considerations. When the current and voltage on a transmission circuit undergo a change from one steady state to another, because of lightning, switching operations, short circuits or other causes, surges are set up which travel along the line with the speed of light. These disturbances are partly reflected and partly passed through at any point where there is a change in the line constants, such for example as a change from overhead to underground construction, at a fork in the line or at the terminals. The waves are attenuated in transit and are damped out in a period of time which, on any practical line, is very small compared with the time constants of the connected machinery. On short lines the traveling wave phenomena are usually over before there is even any appreciable change in the value of the impressed alternating voltage:

It is not easy for one whose knowledge of electricity has been gained by taking up in turn Ohm's law and the hydraulic analogy of current flow, the magnetic circuit, the electrostatic circuit, etc., to visualize the physical mechanism of a traveling wave. He has grown accustomed to thinking of current as flowing only in wires, except in the cases of vacuum tubes, corona or electrolytic conduction. None of these last three being present, as a usual thing, on transmission lines, his impulse is to think of the current as being confined strictly to the wires. This assumption does indeed hold almost exactly for ordinary steady-state operation, but at the front of a traveling wave the analysis is completely inadequate unless the displacement current is considered.

The chief single contribution which Maxwell made to electromagnetic theory was in the introduction of the idea of the displacement current. " The motion of electrons in non-conducting bodies, such as glass and sulphur, kept by the elastic force within certain bounds, *together with the change of the dielectric displacement of the ether itself*, constitutes what Maxwell called the displacement current."[1] According to modern views, both parts of the displacement current cause magnetic effects identical with those of ordinary conduction current. The displacement current of the ether itself is independent of any motion of elec-

[1] Lorentz, *Theory of Electrons*, p. 9.

trons. This is one of the foundations for the electromagnetic theory of light. The fact that wave phenomena which occur in air and in vacuo are nearly identical affords good reason to believe that the displacement current accompanying a traveling wave along an overhead transmission line is independent of any motion of the electrons in the air.

Maxwell frequently makes the statement that electricity acts like an incompressible fluid. When he says this, he does not mean *nearly* incompressible, in the way that steel or rock or water is nearly incompressible, but *absolutely* incompressible. This idea is of considerable value in visualizing the progress of a traveling wave, because, if electricity is like an incompressible fluid, its flow must always be **completely circuital**. In a two-wire line with suddenly impressed voltage the current flows out along one wire and back through the other. In the first period of the transient, before the disturbance has reached the other end, the dielectric between the wires is carrying just as much current as the wires themselves. The direction of the displacement current is from one wire to the other.

In many elementary textbooks the statement may be found that the electrons in a conductor act like an incompressible fluid. This shows an incomplete grasp of Maxwell's ideas, because it is only when the ether displacement also is considered that electricity really resembles an incompressible fluid. More electrons can be added to a wire, and are added whenever it is given a negative charge. It is true that the percentage change in number is very minute, but that is beside the question.

As an illustration of the inadequacy of the simple wire-conduction theory, consider the calculation of inductance. Inductance is calculated by the summing up of fluxes, and the flux density may be found by integrating the infinitesimal magnetizing forces set up by all the infinitesimal current elements. Experiment shows that at the front of a nearly rectangular traveling wave, there may be no current on one side, and a nearly uniform current in the wire on the other. The inductance there as calculated on the basis of wire currents only would be just one-half the true value. The difference is made up by the effect of the displacement current. There is much the same situation as regards capacitance. More complete analysis, taking into account the effect of the displacement current, will show that the capacitance per unit length and the inductance per unit length (skin effect, radiation effects and end effects neglected) may be taken to be the same as for steady operation.

The complete solution for traveling waves of the simplest types has not yet been accomplished. Little advance in the knowledge of the behavior of the wave front has been brought forward since the work of Heaviside. The only analyses available, practically all of which were worked out by Heaviside, proceed on the assumption of fixed values of the line parameters. At the front of a steep wave the penetration of current into the wire must be slight, owing to skin effect, and the effective resistance must be increased very considerably. There is also some decrease in inductance for the same reason. Analysis on the basis of assumed constant line parameters indicates a wave front which remains rectangular as it travels along the line, provided that it is set up initially as a rectangular wave. Actually we know from oscillograms taken on overhead lines, as well as from a qualitative study of the more complete theory including transient skin effect, that a wave initially rectangular will lose this characteristic immediately and develop a more and more rounded front as it travels along a line.

71. The Differential Equations. The differential equations for traveling waves are the same as those already considered for the steady-state alternating-current distribution on a long line, except that the more general $\frac{\partial}{\partial t}$ must be used in place of $j\omega$. We have then, following (174) and (175),

$$\frac{\partial e}{\partial x} = ri + l\frac{\partial i}{\partial t}; \tag{315}$$

$$\frac{\partial i}{\partial x} = ge + c\frac{\partial e}{\partial t}. \tag{316}$$

Eliminating i to obtain an equation in e, x and t, and then eliminating e to obtain an equation in i, x and t, there results:

$$\frac{\partial^2 e}{\partial x^2} = lc\frac{\partial^2 e}{\partial t^2} + (rc + lg)\frac{\partial e}{\partial t} + rge; \tag{317}$$

$$\frac{\partial^2 i}{\partial x^2} = lc\frac{\partial^2 i}{\partial t^2} + (rc + lg)\frac{\partial i}{\partial t} + rgi. \tag{318}$$

These are the most general equations premised on assumed constant line parameters. Actually r varies considerably owing to skin effect at the front of a steep wave; l varies to a much smaller degree. If there is corona on the line, which is the usual condition during lightning transients, the value of the leakance g is highly variable, and the capacitance c may also be considered as variable, although this is largely a matter of definition.

If the leakance is negligible, as is usual on power lines unless corona is present, the two equations (317) and (318) reduce to

$$\frac{\partial^2 e}{\partial x^2} = lc\frac{\partial^2 e}{\partial t^2} + rc\frac{\partial e}{\partial t}; \tag{319}$$

$$\frac{\partial^2 i}{\partial x^2} = lc\frac{\partial^2 i}{\partial t^2} + rc\frac{\partial i}{\partial t}. \tag{320}$$

A special case of peculiar interest, simplicity and importance occurs when the four parameters fulfill the relation

$$\frac{r}{l} = \frac{g}{c}, \tag{321}$$

when the line is said to be "distortionless." An approach to this condition by the addition of inductance, as pointed out first by Oliver Heaviside, greatly improves the performance of communication lines. The resulting differential equations are:

$$\frac{\partial^2 e}{\partial x^2} = lc\frac{\partial^2 e}{\partial t^2} + 2\,rc\frac{\partial e}{\partial t} + rge; \tag{322}$$

$$\frac{\partial^2 i}{\partial x^2} = lc\frac{\partial^2 i}{\partial t^2} + 2\,rc\frac{\partial i}{\partial t} + rgi. \tag{323}$$

The simplest case of all occurs when both resistance and leakance are neglected. The equations then become:

$$\frac{\partial^2 e}{\partial x^2} = lc\frac{\partial^2 e}{\partial t^2}; \tag{324}$$

$$\frac{\partial^2 i}{\partial x^2} = lc\frac{\partial^2 i}{\partial t^2}. \tag{325}$$

Complete consideration of the solutions of all the foregoing differential equations is sufficiently lengthy and complicated to form the subject of a separate book in itself. For the engineer who is interested chiefly in practical understanding of physical phenomena, the more general equations do not offer much more attraction than the simplest, regardless of considerations of time and difficulty, because even (317) and (318) leave out of account such important effects. The danger from traveling waves is most pronounced in the early stages, before they become much attenuated, and in these early stages the simple equations (324) and (325) give almost as good a solution as the most general ones.

72. Voltage Waves on Lossless Line. If we write $lc = 1/v^2$, equation (324) becomes d'Alembert's equation

$$\frac{\partial^2 e}{\partial t^2} - v^2\frac{\partial^2 e}{\partial x^2} = 0. \tag{326}$$

The differential equation of the characteristics[1] is

$$dx^2 - v^2\,dt^2 = 0. \tag{327}$$

This may be factored as $(dx + v\,dt)\,(dx - v\,dt) = 0$, and the solutions are

$$x - vt = \text{constant},$$
$$x + vt = \text{constant}.$$

Equation (326) is reduced to

$$\frac{\partial^2 e}{\partial(x - vt)\,\partial(x + vt)} = 0, \tag{328}$$

and its general solution is

$$e = F(x - vt) + G(x + vt), \tag{329}$$

where F and G are any arbitrary functions. Their forms are determined by initial and boundary conditions. The solution for the current, i, will of course be of the same form as (329), since the differential equations for voltage and current are similar.

The function $F(x - vt)$ is a function which moves to the right (that is, toward larger values of x) with a velocity v, without attenuation or change of form. This is obvious, because, if x and vt are increased equally, the argument $x - vt$ remains unchanged, and so the function of the argument is unchanged.

Direct-current switching transient on short-circuited line. Let us consider first the particular case of an initially uncharged line, with its receiving end grounded, and a direct voltage suddenly applied to the sending end.

The initial conditions are

$$e = 0 \quad \text{for} \quad 0 < x < l \quad \text{when} \quad t = 0; \tag{330}$$
$$i = 0 \quad \text{for} \quad 0 < x < l \quad \text{when} \quad t = 0. \tag{331}$$

The boundary conditions are

$$e = E \quad \text{at} \quad x = 0 \quad \text{for} \quad t > 0; \tag{332}$$
$$e = 0 \quad \text{at} \quad x = l. \tag{333}$$

The origin is taken at the sending end.

From (331) and the relation that $\dfrac{\partial i}{\partial x} = c\,\dfrac{\partial e}{\partial t}$, this initial condition can be expressed in terms of voltage as

$$\frac{\partial e}{\partial t} = 0 \quad \text{for} \quad 0 < x < l \quad \text{when} \quad t = 0. \tag{334}$$

[1] See, for example, Bateman, *Differential Equations.*

Our general equation is

$$e = F(x - vt) + G(x + vt), \tag{335}$$

and upon substituting condition (330) this becomes

$$F(x) + G(x) = 0 \quad \text{for} \quad 0 < x < l. \tag{336}$$

$F(x)$ may be any distribution initially, provided that $G(x)$ is the negative of it, and this condition will be fulfilled.

Substituting condition (334) into the derivative of (335) with respect to t, we find

$$\begin{aligned} \frac{\partial e}{\partial t} &= \frac{\partial}{\partial t}[F(x - vt) + G(x + vt)] \\ &= -vF'(x - vt) + vG'(x + vt) \\ &= -vF'(x) + vG'(x) \quad \text{when} \quad t = 0. \end{aligned} \tag{337}$$

$$F'(x) = G'(x) \quad \text{for} \quad 0 < x < l \quad \text{at} \quad t = 0. \tag{338}$$

The primes denote the derivatives with respect to the arguments.

The F and G functions must have their slopes equal to each other, as well as having the values of the functions themselves equal but of opposite sign. For two curved lines to have equal and opposite ordinates, their slopes must also be equal and opposite. From (338), however, the slopes must be equal, and this combination of conditions can be satisfied only if the slopes are zero. In other words, the functions have flat tops over the range specified, $0 < x < l$.

There is nothing to tell us the actual magnitude of the functions, but since we are interested only in finding their sum, this does not matter. It will serve to assume arbitrarily that $F(x) = 0$ over the line when $t = 0$, and it follows from this that $G(x) = 0$ also.

The boundary conditions (332) and (333) may be substituted in the general equation (335), giving

$$E = F(-vt) + G(vt); \tag{339}$$

$$0 = F(l - vt) + G(l + vt). \tag{340}$$

From (340) it is seen that the G function of an argument which is greater than l by any amount vt is the negative of the F function of the same amount less than l. Since it is known that $F(x) = 0$ in the range 0 to l it follows that $G(x) = 0$ in the range l to $2\,l$. From (339) it appears that the F function of any negative argument $-vt$ is equal to E minus the G function of the same positive argument $+vt$. Since $G(x) = 0$ over the range 0 to $2\,l$, it appears that $F(x) = E$ over the range $-2\,l$ to 0. Now condition (340) can be applied again to show that $G(x) = -E$

from 2 l to 4 l: following that, a reapplication of (339) indicates that $F(x) = 2\,E$ over the range from $-4\,l$ to $-2\,l$, and so on. A plot of the two functions for $t = 0$ is shown in figure 72. The distribution of voltage

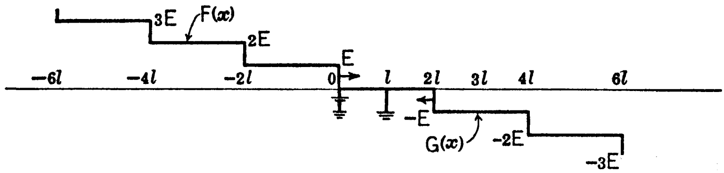


FIG. 72. The functions $F(x)$ and $G(x)$ for the voltage waves on a short-circuited line with direct voltage impressed.

on the line at any time t is determined by moving the $F(x)$ function a distance vt to the right, and the $G(x)$ function a distance vt to the left. The sum of the two functions over the range of x from 0 to l is the indicated voltage distribution. A number of successive distributions at intervals of 0.5 l/v are shown in figure 73.

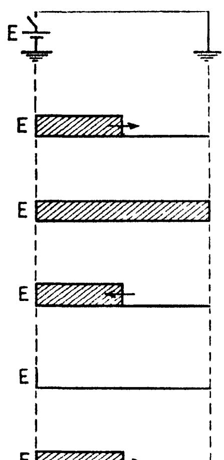


FIG. 73. Voltage waves on a grounded line due to suddenly impressed direct voltage. Effect of resistance neglected.

Direct-current switching transient on open line. For an open line, the same initial conditions exist as in the previous case;

$$e = 0 \quad \text{for} \quad 0 < x < l \quad \text{when} \quad t = 0; \qquad (330)$$

$$\frac{\partial e}{\partial t} = 0 \quad \text{for} \quad 0 < x < l \quad \text{when} \quad t = 0. \qquad (334)$$

The boundary condition at the sending end is the same,

$$e = E \quad \text{at} \quad x = 0 \quad \text{for} \quad t > 0; \qquad (332)$$

but at the open end the condition is

$$i = 0 \quad \text{at} \quad x = l, \qquad (341)$$

which, from the relation that $\frac{\partial e}{\partial x} = l\,\frac{\partial i}{\partial t}$ may be written

$$\frac{\partial e}{\partial x} = 0 \quad \text{at} \quad x = l. \qquad (342)$$

From the standpoint of the engineer or physicist, it will avoid considerable mathematical work without any essential loss of generality to consider that the voltage is impressed on the line not instantly, but in a very short interval of time. The wave fronts will then have finite slopes, and distinction can readily be made between positive and negative slopes. In what follows, this assumption is made.

Just as in the short-circuit case, the $F(x)$ and $G(x)$ functions are flat-topped, and may be taken as equal to zero over the range from 0 to l. Substituting (342) into the general equation gives

$$\frac{\partial e}{\partial x} = \frac{\partial}{\partial t}[F(x - vt) + G(x + vt)]$$

$$= F'(x - vt) + G'(x + vt). \quad (343)$$

$$0 = F'(l - vt) + G'(l + vt). \quad (344)$$

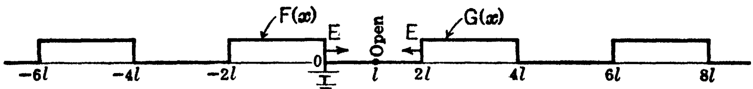


FIG. 74. The functions $F(x)$ and $G(x)$ for the voltage waves on an open line with direct voltage impressed.

It is indicated that the slope of the $G(x)$ function at any distance vt to the right of point l is equal to the negative of the slope of the $F(x)$ function the same distance vt to the left of point l. Since the slope of the $F(x)$ function in this range is zero, the $G(x)$ function may be extended with zero slope as far out as $x = 2\,l$. Now using condition (332) as in the previous case, it is seen that $F(x) = E$ over the range from $-2\,l$ to 0. It goes from 0 to E at or near the point $x = 0$, but it is assumed, and in fact physically necessary, that this takes a finite small time so that the slope is finite. Now applying condition (344) again, it is seen that the $G(x)$ function must slope *up* from left to right at or near $x = 2\,l$ just as $F(x)$ slopes *down* from left to right at or near $x = 0$. Consequently the $G(x)$ function will attain the same height, E, just beyond $x = 2\,l$. Thereafter and out to $4\,l$, the slope will be zero. Now (332) may be used again, followed by (344), and so on. A plot of the $F(x)$ and $G(x)$ functions for $t = 0$ is shown in figure 74. The corresponding voltage distributions at successive intervals of 0.5 l/v are shown in figure 75.

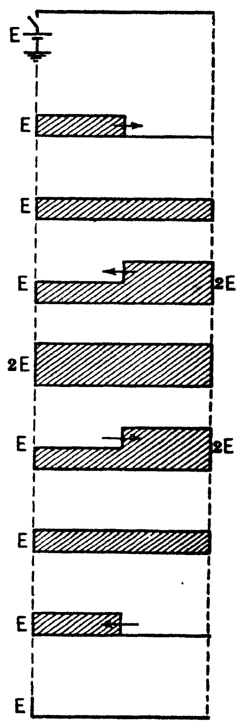


FIG. 75. Voltage waves on an open line due to suddenly impressed direct voltage. Effect of resistance neglected.

73. Current Waves. The waves of current accompanying the waves of voltage may be calculated in a similar way from the original differential equations by eliminating e instead of i, but it is easier, once the voltage solution is obtained, to make use of it in solving for the current. During the first period of the transient on both the open line and the short-circuited line the voltage was changing from 0 to E at a linear speed of $1/\sqrt{lc}$. The charge therefore was increasing at a rate of $cE/\sqrt{lc} = E\sqrt{c/l}$ amperes. The quantity $\sqrt{l/c}$ is called the surge impedance. The current after the various reflections may readily be found by analogous methods, and the details will not be given here since the solution can be written almost by inspection. The reflections are shown in figure 76 (short-circuited line) and figure 77 (open line).

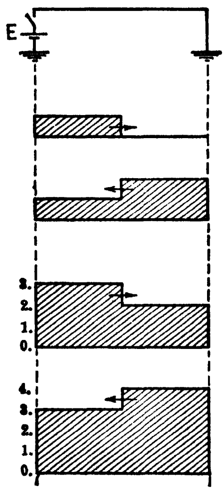


FIG. 76. Current waves on a grounded line due to suddenly impressed direct voltage. Effect of resistance neglected. An ordinate of unity corresponds to a current of $E\sqrt{\frac{c}{l}}$ amperes.

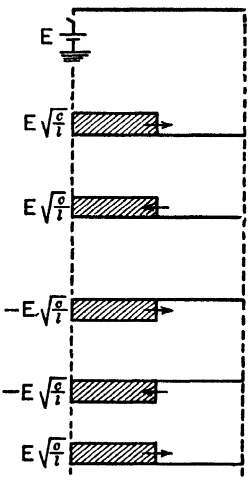


FIG. 77. Current waves on an open line due to suddenly impressed direct voltage. Effect of resistance neglected.

A very useful generality can be evolved from the solutions which have been carried through. A voltage wave, upon striking a short-circuited end, is reflected with a change of sign; upon striking an open end, it is reflected without a change of sign. For current waves, it will be found that the reverse is true. A current wave, upon striking a short-circuited end, is reflected without a change of sign; upon striking an open end, it is reflected with a change of sign.

74. Reflections at Junctions. If a wave of voltage is traveling down a line and arrives at a junction where the original line joins another of different constants, or where it forks out into two or more branches, the original equations still hold, the boundary conditions at the junction being that of continuity of voltage from one section to another, and that of conservation of charge (Kirchhoff's first law).

Consider first the phenomena which takes place when a wave strikes the junction of two lines of different constants. Suppose that a wave of voltage of height E is coming from the left along a previously uncharged line of constants l henries and c farads per unit length, accompanied, necessarily, by a current wave of height $E\sqrt{c/l}$, while the rest of the line on the right of the junction has constants l_1 henries and c_1 farads per unit length.

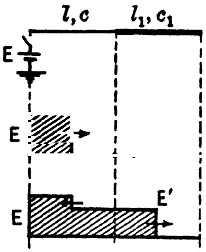


FIG. 78. Partial reflection and partial transmission of voltage wave at a junction. Effect of resistance neglected.

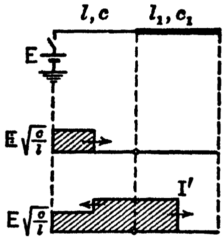


FIG. 79. Partial reflection and partial transmission of current wave at a junction. Effect of resistance neglected.

The general phenomena of partial reflection and partial transmission are shown in figure 78 (voltage wave) and figure 79 (current wave). In order to find out what the heights of the new portions of the wave are, let them be designated by E' and I'. We know that

$$I' = E'\sqrt{\frac{c_1}{l_1}}. \tag{345}$$

$$E\sqrt{\frac{c}{l}} - E'\sqrt{\frac{c_1}{l_1}} = (E' - E)\sqrt{\frac{c}{l}}. \tag{346}$$

Solving (346),

$$E' = \frac{2E}{1 + \sqrt{\frac{l}{c}\frac{c_1}{l_1}}} = E\frac{2\sqrt{\frac{c}{l}}}{\sqrt{\frac{c}{l}} + \sqrt{\frac{c_1}{l_1}}} \tag{347}$$

Substituting this value in (345) gives

$$I' = \frac{2E}{\sqrt{\frac{l_1}{c_1}} + \sqrt{\frac{l}{c}}}. \tag{348}$$

Partial reflections of this sort may in certain cases result in the building up of dangerously high voltages, especially where there is a very marked change in the line constants, such for instance as would occur where a line is run underground for part of the distance and overhead for the remainder.

Forked lines. Consider now the action when a wave of voltage E is coming down a line of constants l and c per unit length, and reaches a fork where the line splits into two or more sections having constants

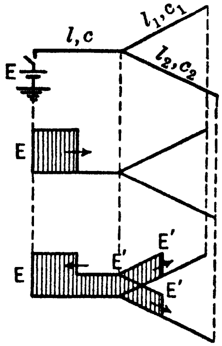


FIG. 80. Partial reflection and partial transmission of a voltage wave at a fork. Effect of resistance neglected.

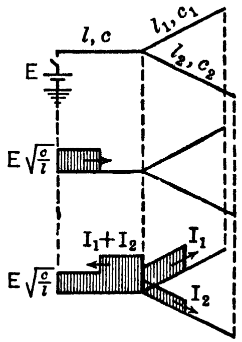


FIG. 81. Partial reflection and partial transmission of a current wave at a fork. Effect of resistance neglected.

per unit length l_1 and c_1, l_2 and c_2, l_3 and c_3, etc. The general form of the phenomena for a bifurcated line is shown in figure 80 (voltage) and figure 81 (current). We know that

$$I_1 = E'\sqrt{\frac{c_1}{l_1}}. \tag{349}$$

$$I_2 = E'\sqrt{\frac{c_2}{l_2}}. \tag{350}$$

$$E\sqrt{\frac{c}{l}} - (E' - E)\sqrt{\frac{c}{l}} = E'\sqrt{\frac{c_1}{l_1}} + E'\sqrt{\frac{c_2}{l_2}}. \tag{351}$$

Solving (351)

$$E' = \frac{2\,E\sqrt{\frac{c}{l}}}{\sqrt{\frac{c}{l}}+\sqrt{\frac{c_1}{l_1}}+\sqrt{\frac{c_2}{l_2}}} \tag{352}$$

The effect on the equations of an increased number of branches of the fork is simply to add more terms to the denominator of (352). The accompanying current waves are given by equations (349) and (350) and others of the same type for additional branches.

Reflection at load. When a traveling wave impinges upon a pure resistance load, the amount reflected can be determined in much the same manner. If the load has any inductance without appreciable distributed capacitance, the action where the wave first strikes is like that at an open end, since the inductance cannot have a current set up in it instantaneously. As a matter of fact, the inductance is always accompanied by some distributed capacitance, although it may be very small. If there is only non-inductive resistance in the load, the voltage there must be equal to the resistance drop. Suppose a voltage wave E to impinge on a resistance load R. Let E' be the subsequent voltage. The initial current wave accompanying E was $E\sqrt{c/l}$. The new current value I' accompanying E' must conform to

$$E' = I'R \tag{353}$$

$$E\sqrt{\frac{c}{l}} - (E' - E)\sqrt{\frac{c}{l}} = I' = \frac{E'}{R}. \tag{354}$$

Solving,

$$E' = \frac{2\,E\sqrt{\frac{c}{l}}}{\sqrt{\frac{c}{l}} + \frac{1}{R}}. \tag{355}$$

If the resistance R is made equal to the surge impedance $\sqrt{l/c}$, there is no reflection, but E remains unchanged all over the line.

If the load contains a series capacitance, it has no effect the first instant, being initially uncharged.

75. Attenuation Due to Resistance and Leakage. The Distortionless Line. We are now ready to study the effects of resistance and leakage, and their damping effect on traveling waves. It would seem, offhand, that after considering the phenomena on lines without losses the next simplest case would be the line having resistance also, but no leakage. However, this is not so, as was pointed out by Heaviside. A very simple solution is available when the four line parameters fulfill the following relation, as has already been pointed out:

$$\frac{r}{l} = \frac{g}{c}. \tag{356}$$

The differential equation (322) for voltage may be written in the following form:

$$\frac{\partial^2 e}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 e}{\partial t^2} + \frac{2\,a}{v}\frac{\partial e}{\partial t} + a^2 e, \tag{357}$$

in which $v = 1/\sqrt{lc}$ and $a = \sqrt{rg}$. Now let

$$e = y\epsilon^{-avt}, \tag{358}$$

where y is a function to be determined. Substituting this value of e in (357) gives

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}, \tag{359}$$

an equation which is identical with (324) except that e is now replaced by y. The solution for y in (359) must be the same as the solution for e in (324), so the solution of (357) must be

$$e = \epsilon^{-rt/l}\{F(x - vt) + G(x + vt)\}. \tag{360}$$

Open line. To determine the behavior of a direct-voltage switching transient on an open-end distortionless line, we may proceed as before to determine that $F(x) = 0$ from 0 to l, and $G(x) = 0$ from 0 to $2\,l$. Substituting the boundary condition at the sending end in (360) gives

$$E = \epsilon^{-avt}\{F(-vt) + G(vt)\}, \tag{361}$$

or

$$F(-vt) = -G(vt) + E\epsilon^{avt}. \tag{362}$$

The value of G over a range of its argument from 0 to $2\,l$ is known to be zero, so

$$F(-vt) = E\epsilon^{avt} \tag{363}$$

or

$$F(-x) = E\epsilon^{ax} \tag{364}$$

over the range for $-x$ from $-2\,l$ to 0. Using the boundary condition at the open end, it may be shown that $G(x)$ for the range $2\,l < x < 4\,l$ is equal to $E\epsilon^{a(x-2\,l)}$. Using (361) again, $F(-x)$ for $-x$ between $-4\,l$ and $-2\,l$ is $E[\epsilon^{-ax} - \epsilon^{-a(x+2\,l)}]$, and so on.

An alternative method of solution is to compute $E\epsilon^{-avt}$ as the distribution on a line of infinite length (for $x < vt$) and then apply this solution to other cases by adding or subtracting reflections as required by the particular connection. After the first passage of the wave down the line, the voltage distribution is this exponential curve as far as $t = l/v$. To calculate the distribution after the second passage, imagine the exponential curve folded back on itself where $t = l/v$. Adding the overlapping part to the part underneath gives the curve of voltage distribution after the second passage. (If the load end had been grounded, the overlapping part would have to be subtracted instead of added.) For the distribution after the third passage, imagine the exponential curve folded again at $t = 2\,l/v$, so that the part between this point and $t = 3\,l/v$ overlaps the other two portions. Subtracting out this third part gives the distribution after the third passage, and so on.

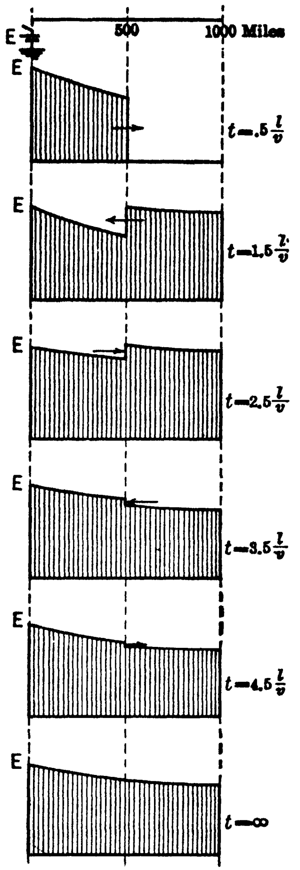


FIG. 82. Successive stages of building up of steady direct voltage distribution on an open distortionless line.

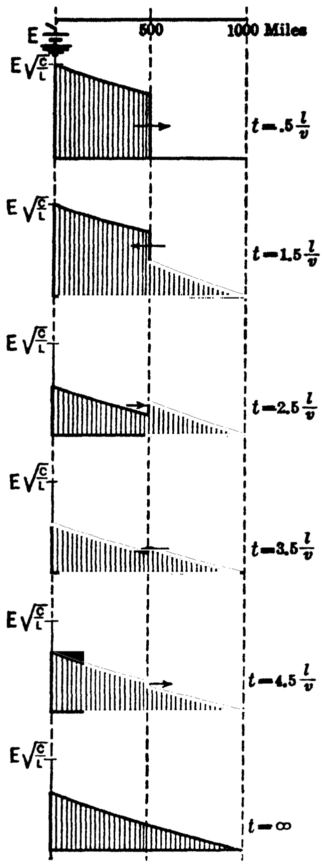


FIG. 83. Successive stages of building up of steady direct current distribution on the open distortionless line of figure 82.

The building up of a direct voltage and current on a two-wire line with leakage sufficient to render the line distortionless is shown in figures 82 and 83. In these figures the first view shows the distribution after the wave had traveled half way down the line. The second view shows the condition after the wave front has traveled half way back to the sending end, and so on. The final views show the steady state distribution. The line constants are:

r = 0.554 ohm per loop mile (no. 0000 copper).

l = 0.00402 henry per loop mile (10-ft spacing, internal inductance neglected).

$c = 7.20 \times 10^{-9}$ farad per loop mile;

$g = 9.88 \times 10^{-7}$ mho per loop mile.

l = 1000 miles.

$a = \sqrt{rg} = 0.000740.$

$\sqrt{l/c}$ = 748 ohms.

$v = \dfrac{1}{\sqrt{lc}}$ = 186 000 miles per second.

The front of the wave is reflected back and forth from one end of the line to the other, being continually attenuated from its initial height E by a factor $\epsilon^{-avt} = \epsilon^{-137.6t}$. The wave of voltage is reflected without a change of sign at the open end, and with a change of sign at the closed (sending) end. The curves of figure 82 show how the voltage is reflected back and forth, and how the transient state finally settles down into the steady state, which may be found from the hyperbolic formula for an open line:

$$e = \frac{E \cosh \alpha(l - x)}{\cosh \theta} \text{ volts } x \text{ miles from sending end.} \qquad (365)$$

The following table gives the values of voltage along the line in terms of E after one, two, three, etc., passages of the wave.

$\frac{vt}{l} = 186\,t$	0	250	500	750	1000 Miles from Gen.
0	1.0000	0	0	0	0
1	1.0000 →	0.8311 →	0.6907 →	0.5741 →	0.4771
2	1.2276 ←	1.1050 ←	1.0203 ←	0.9706 ←	0.9542
3	1.0000 →	0.9158 →	0.8631 →	0.8399 →	0.8456
4	0.9482 ←	0.8528 ←	0.7881 ←	0.7487 ←	0.7370
5	1.0000 →	0.8959 →	0.8239 →	0.7784 →	0.7617
6	1.0118 ←	0.9101 ←	0.8410 ←	0.7989 ←	0.7864
7	1.0000 →	0.9003 →	0.8329 →	0.7921 →	0.7808
∞	1.0000	0.9001	0.8311	0.7907	0.7773

The current wave starts out with a wave front of height $E/748 = 0.001337\,E$. (748 is the surge impedance, equal to $\sqrt{l/c}$ ohms.) It has the same attenuation factor as the voltage, but at the open end it is reflected with a change of sign. The process of its building up to the steady-state distribution is shown by the different curves of figure 83, in which the first curve is the distribution after one passage of the wave, the second one the distribution after two passages, and so on, the last curve showing the steady distribution as given by the hyperbolic formula for open lines,

$$i = \frac{E \sinh \alpha(l - x)}{R_0 \cosh \theta} \text{ amp } x \text{ miles from sending end.} \qquad (366)$$

The following table gives the values of current along the line in terms of $E\sqrt{c/l} = E/748$, the initial current, after one, two, three, etc., passages of the wave.

$\frac{vt}{l} = 186\,t$	Current in Terms of $E\sqrt{\frac{c}{l}} = \frac{E}{748}$				
	0	250	500	750	1000 Miles from Gen.
0	1	0	0	0	0
1	1 0000 →	0.8311 →	0.6907 →	0.5741 →	0.4771
2	0.7724 ←	0.5572 ←	0.3611 ←	0.1776 ←	0
3	0.5448 →	0.3680 →	0.2039 →	0.0469 →	−0.1086
4	0.5936 ←	0.4310 ←	0.2789 ←	0.1381 ←	0
5	0.6454 →	0.4641 →	0.3147 →	0.1678 →	0.0247
6	0.6336 ←	0.4599 ←	0.2946 ←	0.1473 ←	0
7	0.6218 →	0.4501 →	0.2865 →	0.1405 →	−0.0056
∞	0.6291	0.4539	0.2942	0.1447	0

The feature which simplifies the solution for transients when $r/l = g/c$ is the fact that the voltage and current at any point remain unchanged except at the times when the wave front passes by. (This statement of course applies only to direct-current transients.) If this relation among the line constants does not hold, a change is taking place in current and voltage all along the line, as soon as the wave has traversed the line the first time.

Short-circuited line. If the far end of the line is short-circuited, the initial current wave is the same as on the open line, but each reflection is with the same sign, so that the current gradually builds up to a considerably higher value than any shown in figure 83. The final current distribution has the same shape as the final voltage distribution on open circuit, as shown in the last curve of figure 82. The wave of voltage on short circuit, on the other hand, is reflected with a change of sign. It gradually settles down to a steady-state distribution identical in shape with the current distribution on open circuit, as shown in the last curve of figure 83.

Loaded line. If the line has an inductive load, there is total reflection the first instant the wave strikes the load end, but the current immediately begins to build up in the load, thus draining some of the charge from the line and lowering the voltage somewhat from the values shown in figure 82 for the open line. The lowering is most marked near the load end. Reflections at resistance loads and at junctions take place just as already calculated for the resistanceless line.

Voltage at open end. It is interesting to plot from the data already derived the variation with time of the current and voltage at some particular point. For the open line, for instance, the open-end voltage varies as shown in figure 84. The same information as is presented by

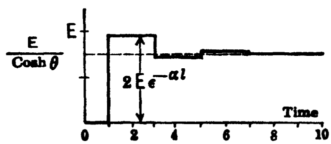


FIG. 84. Time variation of open-end voltage on the open distortionless line of figure 82. Unity on the time scale is equal to the time required for the wave to travel the length of the line.

this curve could be obtained from the series of curves of figure 82 or from the table. However, it is sometimes desirable to have the information presented by a single curve.

There is no voltage at the load until $t = l/v = 1/186$ second after closing the switch. Then according to the theory it suddenly jumps to a value $2 \times 0.4771\ E$, which remains constant for $2\ l/v$ seconds; that is, for the time required for the wave front to travel back to the sending end and return to the open end. The changes as calculated are given in the following table:

t	Voltage at Load in Terms of E
0 to $\frac{l}{v}$	0
$\frac{l}{v}$ to $\frac{3\,l}{v}$	0.9542
$\frac{3\,l}{v}$ to $\frac{5\,l}{v}$	0.7370
$\frac{5\,l}{v}$ to $\frac{7\,l}{v}$	0.7864
$\frac{7\,l}{v}$ to $\frac{9\,l}{v}$	0.7752
∞	0.7773

The generator current variation is shown in figure 85. The numerical data from which this was plotted are:

t	Generator Current in Terms of $E\sqrt{c/l}$
0 to $\frac{2l}{v}$	1.0000
$\frac{2l}{v}$ to $\frac{4l}{v}$	0.5418
$\frac{4l}{v}$ to $\frac{6l}{v}$	0.6454
$\frac{6l}{v}$ to $\frac{8l}{v}$	0.6218
∞	0.6291

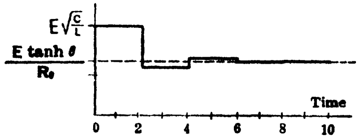


FIG. 85. Time variation of sending-end current on the open distortionless line of figure 82. Unity on the time scale is equal to the time required for the wave to travel the length of the line.

76. General Solutions. For the reasons given near the beginning of this chapter, the derivation of the wave solution for the general cases will not be treated in this book. For purposes of reference, however, the final formulas, in the most convenient forms for computation, are given.

Line with finite resistance, inductance and capacitance. The current wave on an infinite line, due to voltage E suddenly impressed at the sending end, is 0 for $x > vt$, and for $x < vt$ it is

$$i = E\sqrt{\frac{c}{l}}\ \epsilon^{-rt/2l}\ I_0\left\{\frac{r}{2\,l}\sqrt{t^2 - \frac{x^2}{v^2}}\right\} \text{amp.} \tag{367}$$

The corresponding voltage wave is

$$e = E\epsilon^{-rt/2l}\ \{I_0(z) + 2\,sI_1(z) + 2\,s^2I_2(z) + 2\,s^3I_3(z) + \cdots\}, \tag{368}$$

where

$$s = \sqrt{\frac{vt - x}{vt + x}}, \tag{369}$$

$$z = \frac{r}{2\,l}\sqrt{t^2 - \frac{x^2}{v^2}}, \tag{370}$$

$$I_n(z) = \frac{z^n}{2^n n!}\left\{1 + \frac{z^2}{2^2 1!(n+1)} + \frac{z^4}{2^4 2!(n+1)\,(n+2)} + \cdots\right\} \tag{371}$$

Values of I_n functions, called modified bessel functions of the first kind, are given in table XX.

Reflections, if complete as at an open or short-circuited end, may be handled in the manner already described.

TABLE XX

Bessel Functions

(From British Association Report of 1896)

x	$I_0(x)$	$I_1(x)$	$I_2(x)$	$I_3(x)$	$I_4(x)$	$I_5(x)$	$I_6(x)$	$I_7(x)$	$I_8(x)$	$I_9(x)$	$I_{10}(x)$	$I_{11}(x)$	x
0.0	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0
0.2	1.01003	0.1005	$0.0^{2}5017$	$0.0^{3}1671$	$0.0^{5}4175$	$0.0^{7}8347$	$0.0^{8}1391$	$0.0^{10}1987$	$0.0^{12}2483$	$0.0^{14}2758$	$0.0^{16}2758$	$0.0^{18}2507$	0.2
0.4	1.04040	0.2040	0.02027	$0.0^{2}1347$	$0.0^{4}6720$	$0.0^{5}2684$	$0.0^{7}8940$	$0.0^{8}2552$	$0.0^{10}6377$	$0.0^{11}1417$	$0.0^{13}2832$	$0.0^{15}5148$	0.4
0.6	1.09205	0.3137	0.04637	$0.0^{2}4602$	$0.0^{3}3436$	$0.0^{4}2056$	$0.0^{5}1026$	$0.0^{7}4388$	$0.0^{8}1644$	$0.0^{10}5473$	$0.0^{11}1641$	$0.0^{13}4471$	0.6
0.8	1.1665	0.4329	0.08435	0.01110	$0.0^{2}1101$	$0.0^{4}8764$	$0.0^{5}5820$	$0.0^{6}3316$	$0.0^{7}1655$	$0.0^{9}7340$	$0.0^{10}2932$	$0.0^{11}1065$	0.8
1.0	1.2661	0.5652	0.1357	0.02217	$0.0^{2}2737$	$0.0^{3}2715$	$0.0^{4}2249$	$0.0^{5}1599$	$0.0^{7}9961$	$0.0^{8}5518$	$0.0^{9}2753$	$0.0^{10}1249$	1.0
1.2	1.3937	0.7147	0.2026	0.03936	$0.0^{2}5801$	$0.0^{3}6879$	$0.0^{4}6821$	$0.0^{5}5809$	$0.0^{6}4335$	$0.0^{7}2879$	$0.0^{8}1722$	$0.0^{10}9365$	1.2
1.4	1.5534	0.8861	0.2875	0.06452	0.01103	$0.0^{2}1519$	$0.0^{3}1752$	$0.0^{4}1737$	$0.0^{5}1510$	$0.0^{6}1168$	$0.0^{8}8138$	$0.0^{9}5160$	1.4
1.6	1.7500	1.0848	0.3940	0.09989	0.01937	$0.0^{2}3036$	$0.0^{3}3987$	$0.0^{4}4506$	$0.0^{5}4467$	$0.0^{6}3942$	$0.0^{7}3136$	$0.0^{8}2270$	1.6
1.8	1.9896	1.3172	0.5260	0.1482	0.03208	$0.0^{2}5625$	$0.0^{3}8280$	$0.0^{3}1050$	$0.0^{4}1168$	$0.0^{5}1157$	$0.0^{6}1034$	$0.0^{8}8409$	1.8
2.0	2.2796	1.5906	0.6889	0.2127	0.05073	$0.0^{2}9826$	$0.0^{2}1600$	$0.0^{3}2246$	$0.0^{4}2770$	$0.0^{5}3044$	$0.0^{6}3017$	$0.0^{7}2722$	2.0
2.2	2.6291	1.9141	0.8891	0.2976	0.07734	0.01637	$0.0^{2}2919$	$0.0^{3}4492$	$0.0^{4}6076$	$0.0^{5}7329$	$0.0^{6}7975$	$0.0^{7}7903$	2.2
2.4	3.0493	2.2981	1.1342	0.4079	0.1145	0.02626	$0.0^{2}5081$	$0.0^{3}8497$	$0.0^{3}1250$	$0.0^{4}1641$	$0.0^{5}1944$	$0.0^{6}2098$	2.4
2.6	3.5533	2.7554	1.4337	0.5496	0.1654	0.04079	$0.0^{2}8505$	$0.0^{2}1534$	$0.0^{3}2437$	$0.0^{4}3456$	$0.0^{5}4426$	$0.0^{6}5163$	2.6
2.8	4.1573	3.3011	1.7994	0.7305	0.2341	0.06169	0.01377	$0.0^{2}2664$	$0.0^{3}4540$	$0.0^{4}6915$	$0.0^{5}9513$	$0.0^{5}1193$	2.8
3.0	4.8808	3.9534	2.2452	0.9598	0.3257	0.09121	0.02168	$0.0^{2}4472$	$0.0^{3}8137$	$0.0^{3}1324$	$0.0^{4}1946$	$0.0^{5}2610$	3.0
3.2	5.7472	4.7343	2.7883	1.2489	0.4466	0.1323	0.03332	$0.0^{2}7295$	$0.0^{2}1410$	$0.0^{3}2439$	$0.0^{4}3816$	$0.0^{5}5446$	3.2
3.4	6.7848	5.6701	3.4495	1.6119	0.6049	0.1886	0.05015	0.01160	$0.0^{2}2373$	$0.0^{3}4347$	$0.0^{4}7205$	$0.0^{4}1090$	3.4
3.6	8.0277	6.7927	4.2540	2.0661	0.8105	0.2651	0.07411	0.01806	$0.0^{2}38$[illegible]3	$0.0^{3}7523$	$0.0^{3}1316$	$0.0^{4}2103$	3.6
3.8	9.5169	8.1404	5.2325	2.6326	1.0758	0.3678	0.1078	0.02755	$0.0^{2}6243$	$0.0^{2}1269$	$0.0^{3}2336$	$0.0^{4}3929$	3.8
4.0	11.3019	9.7595	6.4222	3.3373	1.4163	0.5047	0.1545	0.04133	$0.0^{2}9810$	$0.0^{2}2090$	$0.0^{3}4038$	$0.0^{4}7131$	4.0
4.2	13.4425	11.7056	7.8684	4.2120	1.8513	0.6857	0.2186	0.06105	0.01514	$0.0^{2}3373$	$0.0^{3}6819$	$0.0^{3}1261$	4.2
4.4	16.0104	14.0462	9.6258	5.2955	2.4046	0.9234	0.3060	0.08894	0.02259	$0.0^{2}5344$	$0.0^{2}1128$	$0.0^{3}2178$	4.4
4.6	19.0926	16.8626	11.7611	6.6355	3.1060	1.2338	0.4239	0.1280	0.03440	$0.0^{2}8324$	$0.0^{2}1830$	$0.0^{3}3653$	4.6
4.8	22.7937	20.2528	14.3550	8.2903	3.9921	1.6369	0.5819	0.1821	0.05080	0.01277	$0.0^{2}2918$	$0.0^{3}6109$	4.8
5.0	27.2399	24.3356	17.5056	10.3312	5.1082	2.1580	0.7923	0.2565	0.07412	0.01932	$0.0^{2}4580$	$0.0^{3}9955$	5.0
5.2	32.5836	29.2543	21.3319	12.8451	6.5106	2.8288	1.0707	0.3580	0.1070	0.02885	$0.0^{2}7086$	$0.0^{2}1596$	5.2
5.4	3[illegible].0088	35.1821	25.9784	15.9388	8.2686	3.6890	1.4371	0.4954	0.1528	0.04260	0.01082	$0.0^{2}2523$	5.4
5.6	46.7376	42.3283	31.6203	19.7424	10.4678	4.7884	1.9171	0.6803	0.2163	0.06222	0.01632	$0.0^{2}3$[illegible]32	5.6
5.8	56.0381	50.9462	38.4704	24.4148	13.2137	6.1890	2.5430	0.9277	0.3037	0.09000	0.02435	$0.0^{2}6052$	5.8
6.0	67.2344	61.3419	46.7871	30.1505	16.6366	7.9685	3.3558	1.2569	0.4230	0.1290	0.03594	$0.0^{2}9207$	6.0

Line with finite r, l, g, c. For the most general case with all four constants finite, the current wave is

$$i = E\sqrt{\frac{c}{l}}\epsilon^{-\rho t}\left\{I_0 + \left[1 - \frac{\sigma}{\rho}\right]d(I_1 + dI_2 - I_3 + d^2I_3 - 2\,dI_4 + I_5 + d^3I_4 - 3\,d^2I_5 + 3\,dI_6 - I_7 + \cdots)\right\}z. \quad (372)$$

$$e = E\epsilon^{-\rho t}\{I_0 + 2\,qsI_1 + (4\,q^2 - 2)s^2I_2 + (8\,q^3 - 6\,q)s^3I_3 + (16\,q^4 - 16\,q^2 + 2)s^4I_4 + (32\,q^5 - 40\,q^3 + 10\,q)s^5I_5 + (64\,q^6 - 96\,q^4 + 36\,q^2 - 2)s^6I_6 + (128\,q^7 - 224\,q^5 + 112\,q^3 - 14\,q)s^7I_7 + (256\,q^8 - 512\,q^6 + 320\,q^4 - 64\,q^2 + 2)s^8I_8 + \cdots\}z. \quad (373)$$

For $x > vt$, $e = i = 0$. In the above, if r, l, c and g are the line parameters per unit length, assumed constant, the meanings of the symbols are:

$$\rho = \frac{r}{2\,l} + \frac{g}{2\,c}$$

$$\sigma = \frac{r}{2\,l} - \frac{g}{2\,c}$$

$$d = \frac{2\,\rho}{\sigma}$$

$$s = \sqrt{\frac{vt - x}{vt + x}}$$

$$q = \frac{\rho}{\sigma}.$$

The argument of all the bessel functions is z, where

$$z = \sigma\sqrt{t^2 - \frac{x^2}{v^2}}.$$

In figure 86 are shown a number of voltage distributions computed by the bessel-function formula (368) for an open line having the following parameters:

Length l = 252 miles.
Inductance l = 4.15 mh per mile.
Capacitance c = 7.17×10^{-9} farad per mile.
Resistance r = 1.004 ohms per mile.
Leakance g = 0.

Computation was made of the distribution on an infinite line for $vt = 0$, $l/4$, $l/2$, $3\,l/4$, l, $5\,l/4$, etc., up to $12\,l$. The first few of these results are given in table XXI.

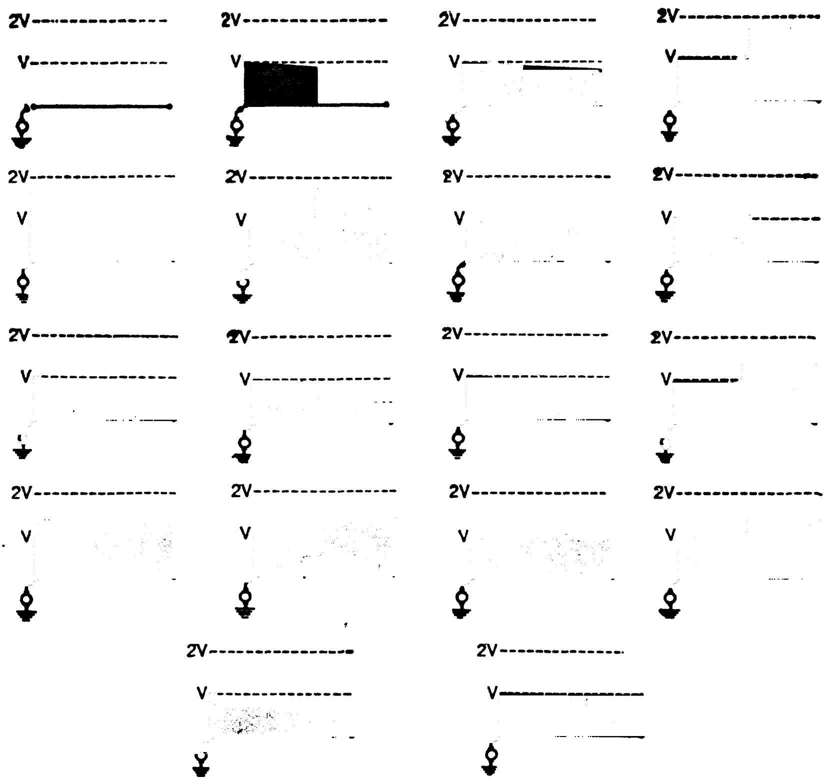


FIG. 86. Computed voltage waves on an open line with assumed constant parameters r, l, c per unit length.

TABLE XXI

VALUES OF COMPUTED VOLTAGE ON INFINITE LINE PER VOLT IMPRESSED

vt (miles)	x (miles)	0	63	126	189	252	315	378
0		1.000	0	0	0	0	0	0
63		1.000	0.959	0	0	0	0	0
126		1.000	0.960	0.920	0	0	0	0
189		1.000	0.9605	0.921	0.883	0	0	0
252		1.000	0.9613	0.923	0.885	0.847	0	
315		1.000	0.962	0.925	0.887	0.850	0.812	0
378		1.000	0.963	0.926	0.889	0.852	0.816	0.779

For the open-line case, the reflection from the end is complete and without change of sign. To illustrate the numerical computation, let $vt = 315$ miles, or 5/4 the length of the line. Then the section of the wave indicated in table XXI as lying between $x = 252$ miles and $x = 315$ miles has actually turned around and gone back from 252 to 189. The wave front is at $x = 189$ and is moving toward the sending end. The distribution is:

x (miles)	e/E
0	1 000
63	0 962
126	0 925
189	0 887 —— 1 699
252	1 700

The first three voltage figures and the first of the two at 189 miles are taken directly from table XXI. The entry at 252 miles is twice the corresponding entry in table XXI. The second entry at 189 miles represents the height of the top of the wave there, and it is the sum of the ordinates of the original wave at $x = 189$ and $x = 315$ miles.

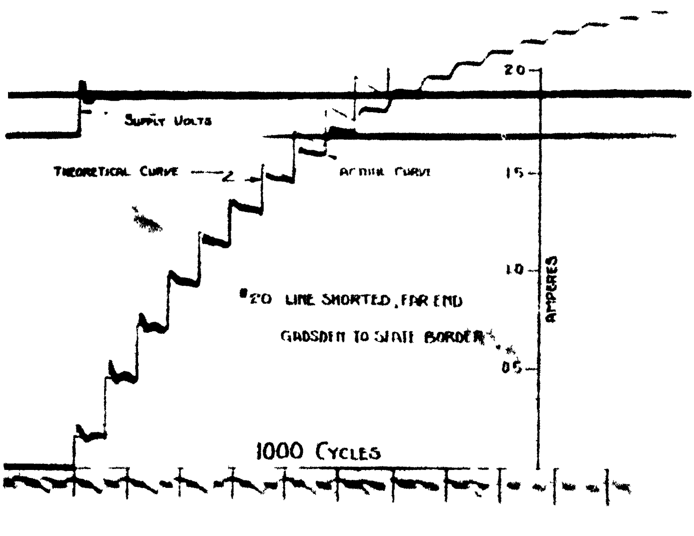


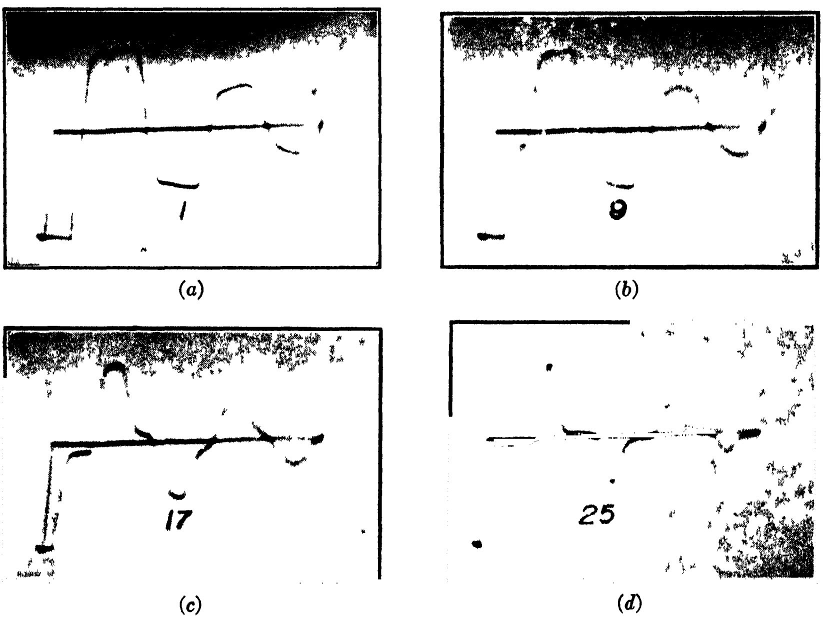
FIG. 87. Oscillogram showing the building up of sending-end current on an overhead transmission line due to a suddenly impressed direct voltage.
E. E. Research Dept., Mass. Inst. of Tech.

77. Experimental Observations of Waves. Figure 87 shows an oscillogram of the sending-end current of a 27-mile overhead transmission line, with a direct voltage suddenly impressed between the two no. 000 copper wires spaced 14 feet apart. A 1000-cps timing wave is shown below, each vertical mark being a half cycle of the wave.

The initial current, about 0.17 ampere, persists nearly unchanged until the wave front has traveled down to the grounded end 27 miles distant, and returned. The current wave is doubled at the grounded end, and upon its return to the sending end undergoes another complete reflection without change of sign, bringing the current up to nearly three times its initial value. It is slightly less than three times on account of the attenuation caused by the line resistance. On the next return of the wave front, the current jumps to nearly five times its original value, then 7, 9, and so on, but with progressively greater attenuation.

There is some overshooting of the experimental curve at the steep parts owing to the inertia of the oscillograph vibrator.

The ruled lines show the computed transient based on assumed constant line parameters r, l and c.



(a) (b) (c) (d)

FIG. 88. Voltage oscillograms on open line with direct voltage impressed.

Artificial smooth lines. Laboratory artificial lines having distributed parameters give results on low-voltage switching transients which are very nearly the same as on corresponding actual lines. A 252-mile length of such a line, having the constants listed in article 76, has been used as the basis for obtaining the data for figures 88 to 96, inclusive.

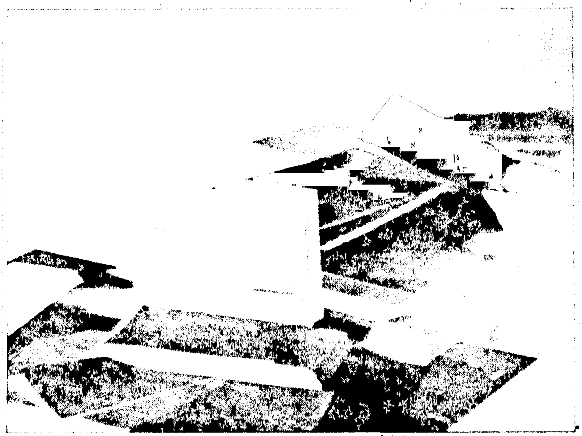


Fig. 89. Cardboard cutouts of voltage-wave distributions on open line, reconstructed from oscillographic data. Direct voltage impressed.

Figure 88 shows four oscillograms of voltage taken on an open line, with direct voltage impressed. The sending-end voltage variation is

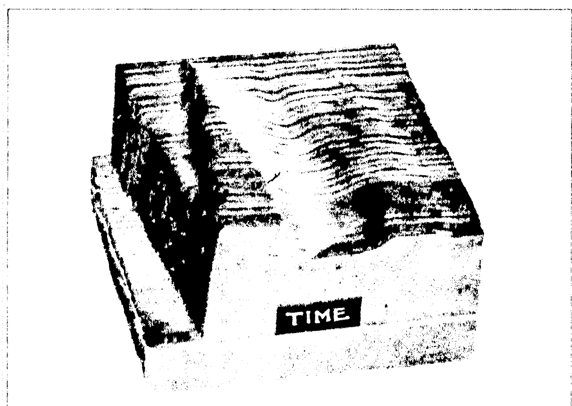


Fig. 90. Variation of voltage with time and distance on a line with a non-inductive load having resistance greater than the surge impedance of the line. Direct voltage impressed.

shown on all four, and the other curves are respectively the voltage at the open end ($x = 252$), and $x = 189$, 126 and 63 miles, in views a, b, c and d.

A large number of cardboard cutouts of voltage distribution for this open line are shown in figure 89. These have been stacked together,

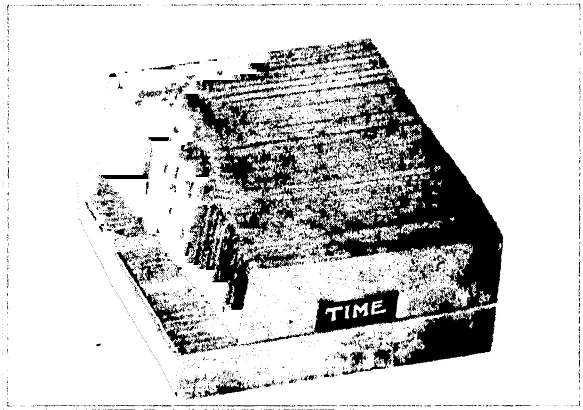


FIG 91 Variation of voltage with time and distance on a line with a non-inductive load having resistance equal to the surge impedance of the line.

forming a three-dimensional model showing the voltage as a function of time and distance.

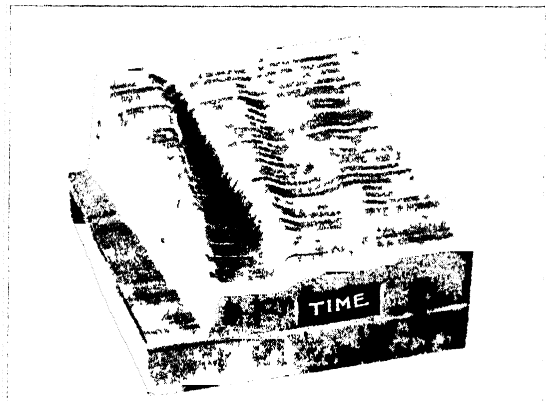


FIG. 92. Variation of voltage with time and distance on a line with a non-inductive load having resistance less than the surge impedance of the line.

A three-dimensional model showing the behavior of a direct-voltage switching wave on a line with a resistance load greater than the surge impedance of the line is shown in figure 90. This model is made of ply-

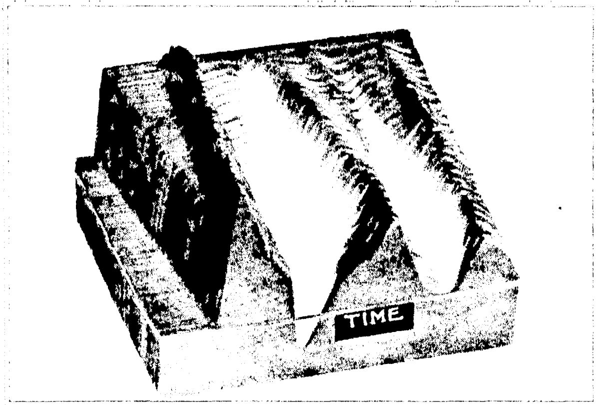


FIG. 93. Direct-voltage waves on a line with inductive load.

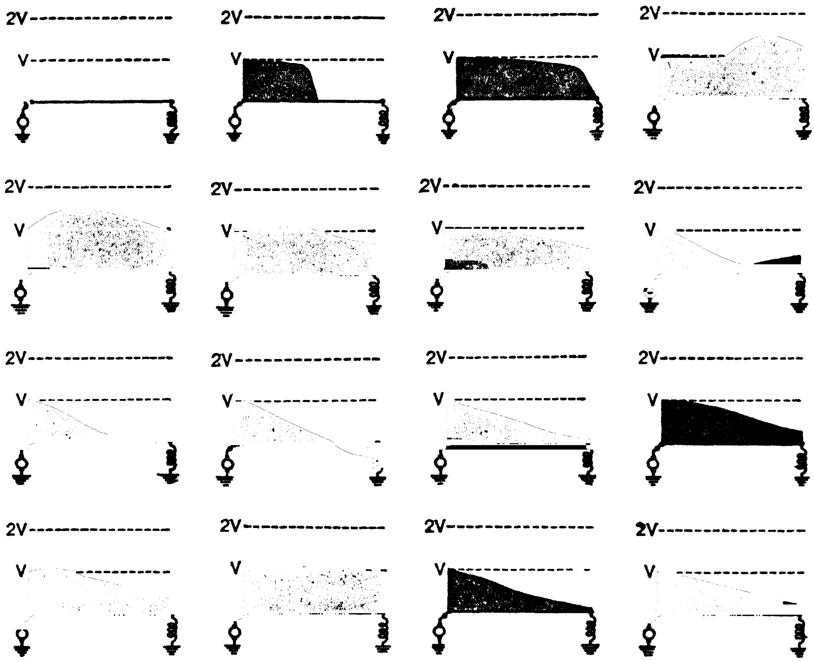


FIG. 94. Voltage waves on d-c line with inductive load The curves are reconstructed from oscillographic data.

wood cutouts the exact shapes of 33 oscillograms taken at equal intervals along the line.

In figure 91 the load resistance has been made equal to the surge impedance.

In figure 92 the load resistance has been made less than the surge impedance.

When the load is inductive, there is initially a complete positive reflection from the load similar to that at an open end, but as the inductance begins to draw current, the charge and voltage on the line near

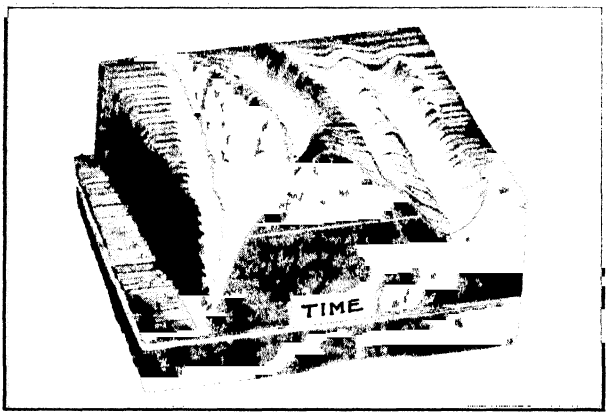


FIG. 95. Direct-voltage waves on a line with capacitive load.

the load are diminished. The entire series of events on such a line, with direct voltage impressed, is shown by the model pictured in figure 93. A series of views of the reconstructed voltage distribution curves are shown in figure 94.

Figure 95 shows the variation of voltage as a function of distance and time on a direct-current line with capacitive load.

78. Alternating-Voltage Waves. On an overhead line 250 miles long it requires only about 1/750 second for a wave to travel the entire length. If the system frequency is 60 cycles per second, this time represents less than a 30-degree phase change. Thus if a switch is closed to energize the line 15 degrees before the maximum voltage is reached, the wave front will reach the far end less than 15 degrees after the maximum. The sending-end voltage would vary only from 0.966 of maximum up to the maximum, and not quite down to 0.966 of maximum, during this interval.

Figure 96*a* shows the variation due to a 60-cps voltage switched on an open line. The rear end of the model gives the variation of the sending-end voltage, which is sinusoidal. The front shows the variation at the open end.

Figure 96*b* is based on a line loaded with a non-inductive resistance greater than the surge impedance of the line.

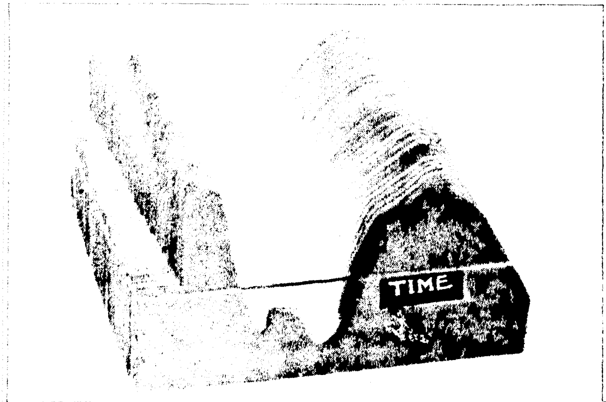


Fig 96*a* Alternating-voltage waves on an open line.

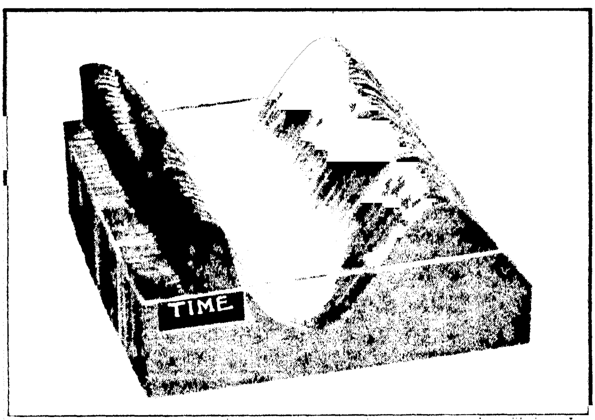


Fig. 96*b* Alternating-voltage waves on a line with non-inductive load having resistance greater than the surge impedance of the line

Alternating-current waves may be computed either by determination of the appropriate F and G functions, or the direct-current solutions may be used in connection with the superposition formula to determine the alternating-current solutions.

79. The Superposition Formula. In any network of linear impedances, if a suddenly impressed unit force at a driving point produces an effect $A(t)$ at a measuring point, then any time variation $f(t)$ of force at the driving point will produce at the measuring point an effect i equal to

$$
\begin{aligned}
i &= \frac{d}{dt}\int_0^t f(t-\lambda)\cdot A(\lambda)\,d\lambda \\
&= \frac{d}{dt}\int_0^t f(\lambda)\cdot A(t-\lambda)\,d\lambda \\
&= A(t)\,f(0) + \int_0^t A(t-\lambda)\,f'(\lambda)\,d\lambda \\
&= A(0)\,f(t) + \int_0^t f(t-\lambda)\,A'(\lambda)\,d\lambda. \qquad (374)
\end{aligned}
$$

Proof. Referring to figures 97 and 98, $A(t)$ is shown as the response to the unit function 1 impressed. The function $f(t)$ may be considered as broken up into a series of very small steps. The height of a step at

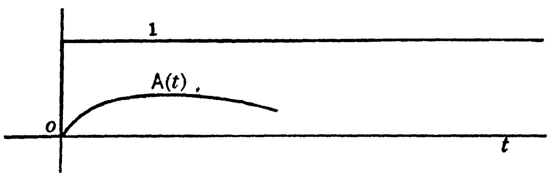


FIG. 97. The unit function 1 and the response function $A(t)$.

point λ on the time axis is $f'(\lambda)\,d\lambda$. The impressing of such a force on the network causes the origination of another response function which is of the form $A(t)$, but which starts at time λ and has a magnitude

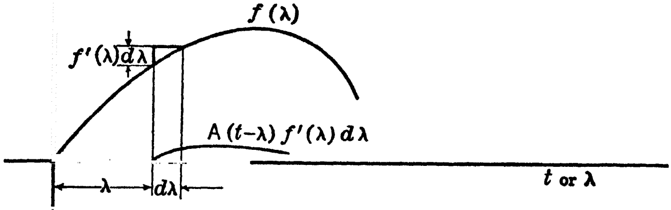


FIG. 98. Variable impressed force $f(\lambda)$, showing treatment as a step function, and response to a step of height $f'(\lambda)\,d\lambda$.

proportional to the force. Its equation is therefore $A(t-\lambda)\,f'(\lambda)\,d\lambda$. If we let $d\lambda$ approach zero and integrate this expression from 0 to t, we have the equation for the actual response to a force $f(t)$, except where $f(t)$ has a finite value when $t = 0$. If this is the case, we must add the response term to correspond, which is $A(t)f(0)$. Thus the third form of the superposition formula given in (374) is obtained. It may be shown that the other forms are equivalent.

If $f(t)$ is sinusoidal, of the form

$$e = E \sin (\omega t + \alpha), \tag{375}$$

the response function may be written

$$\begin{aligned} i &= A(0)\, E \sin (\omega t + \alpha) \\ &+ E \sin (\omega t + \alpha) \int_0^t \cos \omega t A'(t)\, dt \\ &- E \cos (\omega t + \alpha) \int_0^t \sin \omega t A'(t)\, dt. \end{aligned} \tag{376}$$

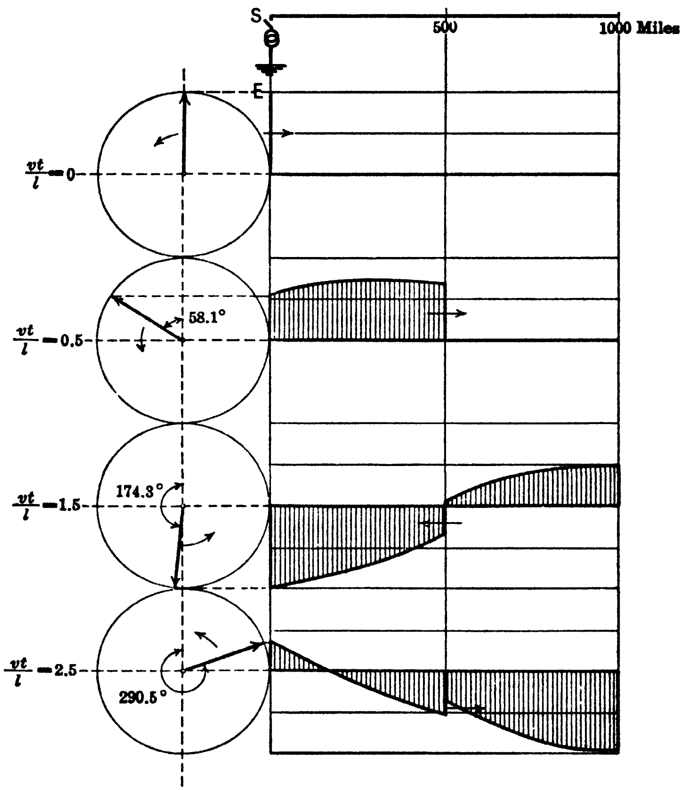


FIG. 99. Successive stages of building up of steady alternating-voltage distribution on a distortionless open line, due to a 60-cps voltage suddenly impressed at the peak of the wave.

The computed alternating-voltage transient with subsidence to steady state may be worked out readily on the basis of the distortionless line. This is illustrated in figure 99, which shows the computed voltage waves based on the open-end line described in article 75.

80. Waves Produced by Lightning. The source of traveling waves which is most dangerous to power system operation is lightning. A lightning stroke making a direct hit on a power conductor of course raises its potential enormously; measurements have been made showing that the order of magnitude may be several million volts. There is great danger that this will flash over the insulators. Protection is secured against direct stroke by the use of ground wires over the power conductors, so placed as to intercept direct strokes and shield the line. These ground wires are grounded at each tower. Further protection is secured by the use of lightning arresters, which act as safety valves in passing excess charge from line conductors to ground, thereafter resuming their insulating character. Very high-voltage waves have rapid attenuation on account of the large corona loss.

If a direct lightning stroke occurs to the ground wire or tower, there may be danger of an insulator flashover owing to the potential drop caused by the heavy lightning current, which may be of the order of 100,000 amperes, flowing through the tower footing resistance. Hence it is important to keep the footing resistance low, and if this cannot be done owing to the character of the soil, it may be necessary to use a counterpoise. A counterpoise consists of wires usually buried a few inches in the ground, radiating out from the base of a tower. These wires are sometimes but not always extended so as to connect adjacent towers. They serve two purposes — one to decrease the resistance to ground, and the other to increase the capacitance of the circuit which includes the tower. For a given charge brought down by lightning, the potential produced, other things being equal, is inversely proportional to the capacitance of the body which receives the charge.

Voltages caused by sudden changes in field are induced on conductors in the vicinity of an electric storm.. These are of much more frequent occurrence than direct strokes, but they are not nearly so severe, and field tests seem to indicate that induced voltages do not ordinarily cause damage to apparatus or interruption to service. If overhead ground wires are used, they serve to shield the line conductors partially from induced voltages, as well as afford protection against direct strokes. They also serve to lead the lightning current to ground through several paths, so that the current in any one tower will be only a fraction of the current of the entire stroke.

Whether the lightning disturbance on a power conductor is caused by induction or by a direct stroke, the wave begins as an approximately stationary charge on the line. Let us consider the theoretical behavior of such a charge producing a potential having an arbitrary initial distribution $\phi(x)$.

Referring to figure 100, the potential distribution is given by the

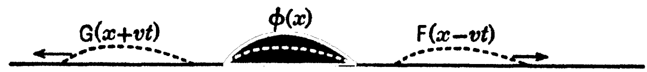


FIG. 100. Bound charge causing a potential distribution $\phi(x)$ on a smooth line.

curve $\phi(x)$. The treatment will be on the basis of the lossless line. The general equation will hold; that is,

$$e = F(x - vt) + G(x + vt). \tag{377}$$

When $t = 0$, $e = \phi(x)$ and $i = 0$. The latter condition can be interpreted as meaning $\frac{\partial e}{\partial t} = 0$, as pointed out earlier. We have then

$$F(x) + G(x) = \phi(x); \tag{378}$$

$$F'(x) = G'(x). \tag{379}$$

If the function $\phi(x)$ is continuous and has continuous derivatives, then $F(x)$ and $G(x)$ can satisfy both these conditions only by each being just one-half of $\phi(x)$, except for an arbitrary constant term which may be added to one and subtracted from the other. Since the F function moves to the right and the G function to the left with velocity v, after t seconds they will have moved out to positions such as those shown in the figure.

Expensive terminal apparatus, like transformers, may be protected by the use of lightning arresters and by special ground-wire protection and special grounding of towers within a few miles of the apparatus. It is important to coordinate the level of insulation for the different elements of the system so that when breakdown does occur it will do minimum damage.

When one or more of a group of multiple conductors carries a surge, secondary surges of smaller amplitude will be induced in the other cond tors.

81. Waves Produced by Circuit Interruption. Voltages of large magnitude may be produced by the sudden interruption of heavy currents. It has already been seen that the ratio between the height of a voltage wave front and the accompanying current wave front is equal to the surge impedance $\sqrt{l/c}$. This quantity for an overhead line is of the order of magnitude of several hundred ohms. Suddenly opening a circuit which carries 1000 amperes might then produce a voltage wave of about 500,000 volts. The formation of an arc in the circuit breaker allows the current to die down relatively slowly, and in alternating-current circuits the tendency is for the arc to persist until the current is passing through a natural zero, and then be extinguished.

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PROBLEMS ON CHAPTER VII

Prob. 1–7. A transmission line runs from the sending end through an underground cable for 15 miles, and then along an overhead line for 20 miles. The end of the line is open. The constants are:

Cable: 1.21×10^{-3} henry per mile,
6.7×10^{-8} farad per mile;
Overhead line: 4.02×10^{-3} henry per mile,
7.2×10^{-9} farad per mile.

A direct voltage of magnitude 1000 is suddenly impressed at the sending end. Compute on the basis of the lossless line, and show on graphs, the voltage distributions at the following times:

(*a*) Just before the wave front reaches the junction.
(*b*) Just after the wave front reaches the junction.
(*c*) Just after the wave has reached the open end.
(*d*) Just after the first reflection has taken place at the sending end.
(*e*) Just after the first reflection has returned to the junction.
(*f*) Just after the second reflection has returned to the junction.

Prob. 2–7. From the junction point in the line of problem 1–7 a second overhead line of the same constants as the first is run a distance of 40 miles, its distant end being open. Show on graphs the voltage distributions at the following instants:

(*a*) Just before the wave front reaches the fork.
(*b*) Just after the wave front reaches the fork.
(*c*) Just after the first reflection at the end of the first fork.
(*d*) Just after the first reflection at the generator.
(*e*) Just after the first reflection at the end of the second fork.
(*f*) Just after the first reflection has arrived at the junction.
(*g*) Just after the second reflection has arrived at the junction.
(*h*) Just after the third reflection has arrived at the junction.

Prob. 3–7. Plot the current distributions in the circuit of problem 1–7.

Prob. 4–7. Plot the current distributions in the circuit of problem 2–7.

Prob. 5–7. Work problem 2–7 with the 40-mile branch grounded at its distant end.

Prob. 6–7. Plot a set of voltage curves similar to the first five curves of figure 86, and for the same line, but with the distant end grounded.

Prob. 7–7. Plot the current distribution curves for the line of problem 6–7.

Prob. 8–7. A 200-mile line having the same constants per mile as the 1000-mile distortionless line described in article 75 has a non-inductive resistance of 500 ohms at the load end, connected between the two line wires. Show by a set of voltage distribution curves the process of building up to the steady voltage distribution.

Prob. 9–7. In problem 8–7, show the process of building up of current by a set of current distribution curves.

Prob. 10–7. Prove the identity of the steady-state hyperbolic function solution for voltage at point x on an open distortionless line with direct voltage impressed, and the solution obtained by summing up the infinite series of reflected waves.

Prob. 11–7. Work problem 10–7 for a short-circuited distortionless line.

Prob. 12–7. Work problem 11–7 for the current.

Prob. 13–7. Calculate and plot current distribution curves corresponding to the voltage curves of figure 99.

CHAPTER VIII

POWER LIMITS AND STABILITY

There is an upper limit, imposed primarily by the voltage drop, to the amount of power which can be transmitted at a given voltage by any transmission line in an alternating-current system. This limit ordinarily marks the operating point at which the synchronous apparatus at the two ends of the line will fall out of synchronism, but this is not necessarily true. There exists a power limit even if there is no synchronous apparatus at one end of the line.

It would not be possible to present anything approaching a thorough treatment of steady-state and transient stability without very greatly increasing the length of this book. This chapter therefore is restricted to a presentation of the general principles only.

82. Power Limit with Static Load. The simplest example of power limit in alternating-current transmission relates to a static impedance load fed by a transmission line on which constant voltage is maintained at the sending end. A plot may be made between receiver voltage and receiver power, similar to figure 101, and the point of maximum power, P_m, for any power factor, is at once apparent. It is marked for each power factor shown.

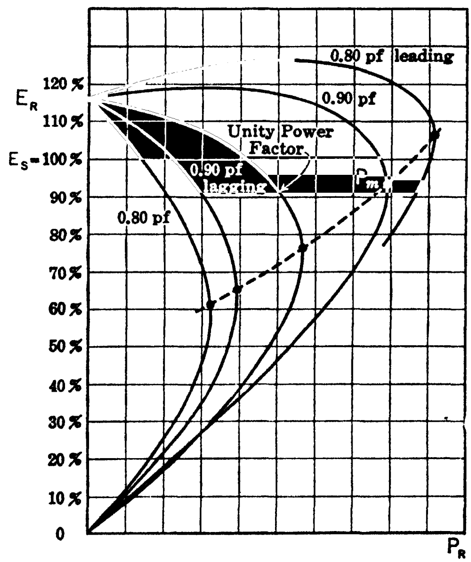

FIG. 101. Receiving-end voltage-power characteristics at different power factors, for fixed (100 per cent) sending-end voltage.

Consider analytically the limit of power which may be delivered through a simple series reactance X by a constant-voltage source E_S. If the load impedance is $Z_L \underline{/\theta_L}$, the power which it receives is

$$\frac{E_S^2 Z_L \cos\theta_L}{(X + Z_L \sin\theta_L)^2 + Z_L^2 \cos^2\theta_L}$$

If θ_L is constant, then by differentiating and equating to zero, it is found that maximum power is received when $Z_L = X$. The indicated maximum is equal to $\dfrac{E_S{}^2 \cos \theta_L}{2\,X\,(1 + \sin \theta_L)}$. The sign of θ_L is positive for an inductive load, negative for a capacitive load.

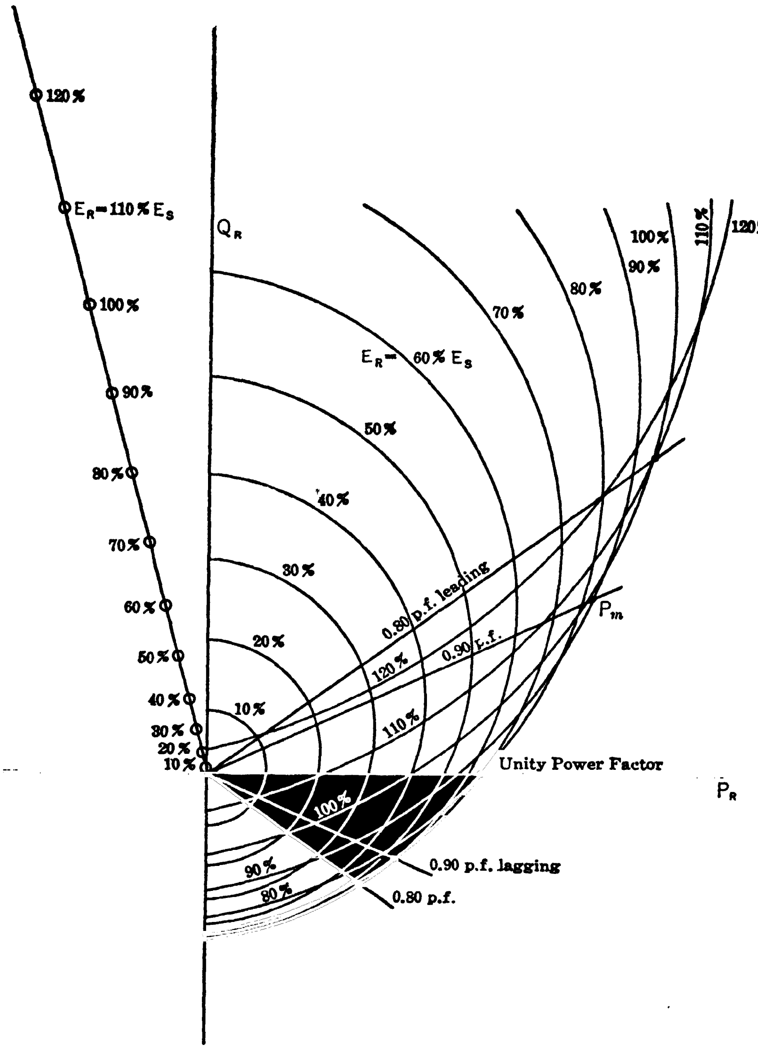

FIG. 102. Receiving-end power chart and power-factor lines.

Another method of determining this power limit is to prepare a chart of receiving-end power, figure 102, based on the fixed sending-end voltage. Then for any given load power factor, the power limit is obviously the abscissa of the point of intersection P_m between the power factor line and the envelope of the family of receiving-end power circles. This must be true because there is no value of receiver voltage which will yield an operating point corresponding to a larger receiving-end power. The maximum power is again marked for each power factor shown.

Either one of these methods serves to determine the power limit imposed by the transmission line alone (including terminal transformers, if their constants are taken into account in determining the general circuit constants on which the diagrams are based).

If the sending-end voltage of the line is not held constant, but the generator or generators are operated with constant field current, then the generator impedances need to be taken into account. If the generator rotors are cylindrical, the synchronous impedance of the machines corresponding to the degree of saturation that prevails may be used and included in the determination of the general circuit constants and circle diagrams. A single equivalent generator may be used as an approximate equivalent to a number of generators in parallel, the impedance of the equivalent machine being made equal to the parallel impedance of the individual machines. If large discrepancies exist between the excitation voltages and percentage impedances of the individual generators, appreciable errors may be introduced by attempting to lump them into a single equivalent machine.

83. Power Limit of a Line of Small Capacity Connecting Two Large Systems. Another simple power-limit problem is that of a line of relatively small capacity connecting two large systems. If the systems are assumed to have infinite capacities, and are both regulated for constant voltage at the points of connection to the line, then the power limit of the line is very nearly equal to the greatest possible abscissa of the receiving-end power circle. This quantity is the radius of the power circle less the abscissa of the circle center, and is equal to

$$\frac{0.003\ E_R}{b}\left[E_S - aE_R \cos\ (\theta_b - \theta_a)\right]\text{kw.} \tag{380}$$

This is the power limit imposed by the characteristics of the line alone to the receiving-end power. The maximum value of sending-end power is

$$\frac{0.003\ E_S}{b}\left[E_R + dE_S \cos\ (\theta_b - \theta_d)\right]\text{kw.} \tag{381}$$

84. Mechanical Analogy to the Stability Problem. S. B. Griscom has suggested a mechanical analog which is very useful in visualizing the stability problem, both steady-state and transient. It is also useful in giving fair quantitative results when its constants are properly adjusted to correspond to the analogous constants of the power system to be studied, although many of the refinements which may be taken into account in calculation cannot be handled on the model.

A sketch of a simple form of the analog, for a two-machine system, is shown in figure 103. Two movable arms are pivoted independently on a common axis. In the hubs of both arms are pulley grooves, and a loop of thread spans them as shown. The thread links a weighted pulley hanging below the shaft of the device. A spring connects the two arms near their upper ends.

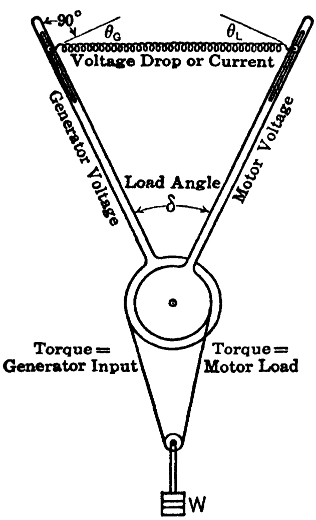

FIG. 103. Mechanical analog for approximate representation of two-machine power system stability characteristics. — *Griscom.*

If the arm lengths are adjusted in proportion to generator and load voltages, then the length of the spring will be equal, to the same scale, to the voltage drop. Since this drop is proportional to the current, the spring length may also be considered as a measure, to another scale, of the size of current.

The torque produced on an arm by the spring tension is equal to the product of the arm length by the tension and by the cosine of the angle θ. In the figure, θ_G is analogous to the generator power-factor angle, and θ_L is analogous to the load or motor power-factor angle. Assuming that the spring's tension is proportional to its length, then the torque is $EI \cos \theta$ or power. If the system is loaded up by adding weights W, the voltage arms will swing stably farther and farther apart until their restoring torque due to the spring, $EI \cos \theta$, begins to diminish with increasing angle δ. This point marks the limit of stability, and the arms fall away from each other. If the arms are of equal length, this occurs when each of the θ's is 45 degrees.

Transient effects of suddenly applied heavy loads may be studied by adding suddenly (but without impact) large increments of weight W. Line faults may be simulated by deflecting the spring toward the pivot. Intermediate condenser stations and other multi-machine problems on a straight-away line may be represented.

85. Effect of Machine Characteristics. If a power-angle curve is prepared, based upon the conditions which actually exist in the system;

the point of instability under steady-state operation will be where the slope $\frac{dP}{d\theta}$ becomes zero, that is, at the maximum power point. The angle θ must be measured, not between the terminal voltages of the transmission line or system, but from the equivalent excitation voltages of the main synchronous machines at the two ends of the line. When a small machine is connected through an impedance to a large system, the point of instability is determined by $\frac{d\theta}{dt}$ becoming zero at the small-machine end; thus the other end may possibly, in theory at least, operate beyond the angle of maximum power.

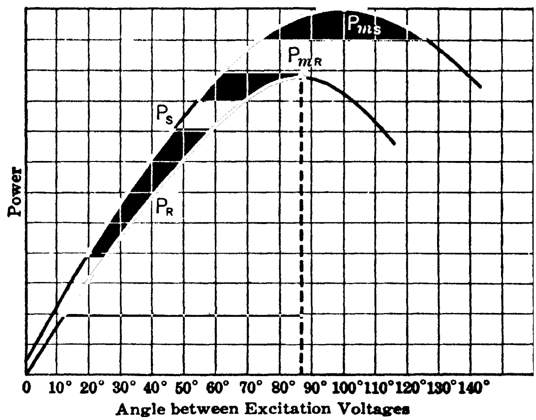

FIG. 104. Power-angle curves.

It may be in certain cases that the capacity of synchronous machinery at one end is so great that no appreciable variation will take place in the angle between the excitation voltage there and the line terminal voltage as the result of any possible change in line power.

It is obvious on consideration that the angle used in the power-angle curves should be that between the rotors, since they are the physical elements which normally operate in synchronism, to which external torques are applied, and which at times will fall out of synchronism.

A typical pair of power-angle curves, one for air-gap power at the generator and the other for the receiver or equivalent receiver, are shown in figure 104. Instability will be reached at the point marked P_{mR}, which is the maximum point on the receiving-end power-angle curve, rather than at P_{mS}, unless the receiving end comprises a large system, and loss of stability in that case would occur at P_{mS} by the generator pulling ahead of the rest of the system as it loses synchronism. These curves are drawn for constant field current.

Terminal voltages at important generating stations and substations are ordinarily controlled by automatic regulators. As the machines are gradually loaded up and approach their stability limit, however, any small increment of load must, if stability is to be maintained, give rise to immediate reactions in the machines which will increase and stabilize the power output at the new level. Ordinary regulators cannot maintain constant terminal voltages during such readjustments, and the stability of the system depends in considerable measure on the quickness of response of the exciter and regulator equipment. The worst condition is to have the field excitation remain practically unchanged during this adjustment period. This condition is the one ordinarily assumed for steady-state stability calculations unless a quick-response excitation system has been provided.

Saturation effects may considerably change the machine parameters at different degrees of excitation, and it is necessary to take this fact into account.

Governor and prime-mover characteristics do not have an important bearing on steady-state stability limits as a general thing, although they need to be considered in a precise study of large transient swings.

The excitation voltages are assumed constant, and acting through the synchronous impedances of the machines (adjusted for the degree of saturation which exists). This affords a most convenient way to compute the stability limit. It is not necessary to compute and plot the whole power-angle curve. The maximum point always occurs near 90 degrees for practical systems.

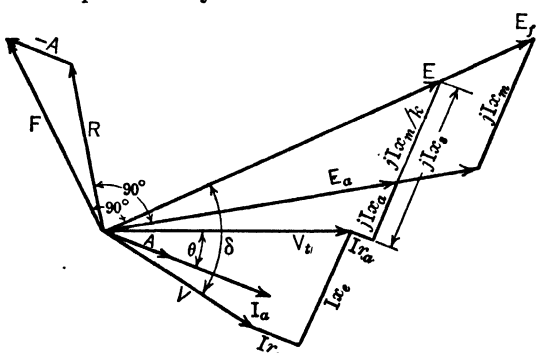

FIG. 105. Vector diagram of a saturated cylindrical-rotor synchronous generator. —*Kingsley.*

Cylindrical-rotor machine. A convenient way of taking into account the effect of saturation on synchronous impedance and stability has been presented by Charles Kingsley, Jr. (see reference at end of chapter), and is reproduced here. This relates to cylindrical-rotor machines such as steam turboalternators.

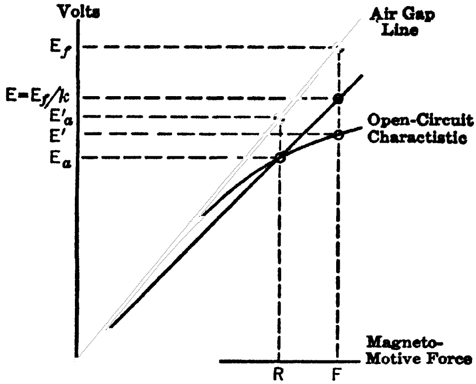

FIG. 106. Open-circuit characteristic. — *Kingsley.*

Consider a generator of this type supplying power to a very large system (considered as an " infinite bus ") through an impedance. The vector diagram is shown in figure 105, and the test characteristics in

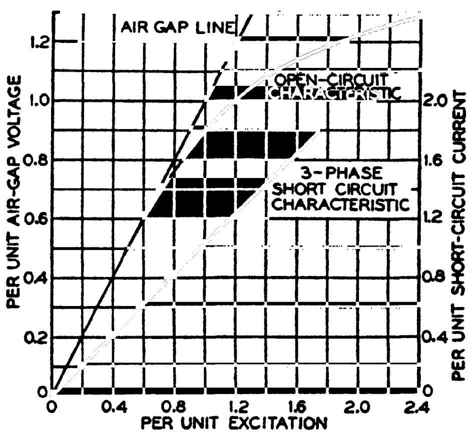

FIG. 107. Open-circuit and short-circuit characteristics of a cylindrical-rotor synchronous machine. — *Kingsley.*

figures 106 and 107. The definitions of the symbols, many of which are shown more clearly in the figures, are:

V_t = terminal voltage (to neutral).

I_a = armature current.

A = mmf due to armature current.

F = mmf due to field current.
r_e = external resistance.
x_e = external reactance.
R = resultant mmf.
r_a = effective armature resistance.
x_a = armature leakage reactance.
x_S = synchronous reactance.
E_a = generated voltage.
E_a' = value generated voltage would have for same R and without saturation.
k = E_a'/E_a, saturation factor.
E = generated voltage due to flux caused by field mmf.
E_f = kE.
δ = load angle between E and V, that is, between rotor of generator and infinite bus.
θ = power-factor angle of machine.
x_d = value of x_S if there is no saturation.
x_m = $x_d - x_a$ = reactance equivalent to armature reaction in the absence of saturation.

$x_S = x_a + \frac{x_m}{k}$.
r = total resistance = $r_e + r_a$.
x = total reactance = $x_e + x_S$.
z = total impedance = $\sqrt{r^2 + x^2}$.
$\alpha = \tan^{-1}\frac{r}{x}$.

If basic test data such as the characteristics of figure 107 together with the value of r_a are available, then for any specified operating condition the vector diagram of figure 105 can be constructed and the synchronous impedance found.

By analogy to equations (309) and (310), we may write the values of generator air-gap power P_a and receiver power P_R, in terms of voltage angle δ, as

$$P_a = \frac{EV}{z}\sin(\delta - \alpha) + \frac{E^2 r}{z^2}$$

$$= \frac{E_f V}{kz}\sin(\delta - \alpha) + \frac{E_f^2 r}{k^2 z^2}; \quad (382)$$

$$P_R = \frac{EV}{z}\sin(\delta + \alpha) - \frac{V^2 r}{z^2}$$

$$= \frac{E_f V}{kz}\sin(\delta + \alpha) - \frac{V^2 r}{z^2}. \quad (383)$$

It may be shown that

$$(kzE_a)^2 = (E_f Z_C)^2 + (Vx_m)^2 + 2\,E_f V Z_C x_m \cos(\delta - \alpha_C), \quad (384)$$

in which

$$Z_C = \sqrt{(r_e + r_a)^2 + (x_e + x_a)^2},$$

$$\alpha_C = \tan^{-1}\frac{r_e + r_a}{x_e + x_a}.$$

From this relation values of α, kz and z^2, for substitution in (382) and (383), may be computed for any assumed values of E_a. The auxiliary curves of figure 109 are thus obtained. From such curves and equations (382) and (383), the power-angle curves may be determined.

If the external circuit is not representable by a simple series impedance $r_e + jx_e$, but requires the use of the general circuit constants or the equivalent π, the sending-end power-angle curve may be determined by a modification of the method just described. Let the external circuit have the general constants A, B, C and D. Then a hypothetical system may be substituted, in which the voltage of the infinite bus is made equal to V/D, and the general circuit is replaced by a simple series impedance equal to B/D. If the original representation is in terms of an equivalent π whose constants are Z, Y_S and Y_R, then in the simplified circuit V becomes $V/(1 + ZY_S)$, and the external circuit becomes a simple series impedance equal to $Z/(1 + ZY_S)$. The voltage at the infinite bus, in the hypothetical systems, is changed from its original value in both phase and magnitude, as indicated in the formulas.

This method does not give directly the receiving-end power, but it may be found by subtracting the losses from the sending-end power. The angle which should be used in the final power-angle curves should be that between the generator and the actual, not the hypothetical, infinite bus.

Example. Using the test data presented in figure 107, calculate the power-angle curves when this machine feeds an infinite bus through an external impedance $0.092 + j\,0.436$ per unit, the armature leakage impedance being $0.022 + j\,0.105$ per unit. $V = 1.00$ and $E_f = 1.61$ on a per-unit basis.

Solution. Values of k and x_S, derived from figure 107, are shown in figure 108. We have:

$$r = r_e + r_a = 0.092 + 0.022 = 0.114.$$

Let $E_a = 1.20$. Then from figure 108, $k = 1.57$ and $x_S = 0.66$. The total reactance x at this degree of saturation is

$$x = x_e + x_S = 0.436 + 0.66 = 1.096.$$
$$z = \sqrt{r^2 + x^2} = 1.10.$$
$$z^2 = 1.21.$$
$$kz = 1.73.$$
$$(kzE_a)^2 = 4.31.$$
$$\alpha = \tan^{-1}\frac{r}{x} = \tan^{-1}\frac{0.114}{1.096} = 5.9°.$$
$$Z_C = \sqrt{0.114^2 + 0.541} = 0.553.$$
$$\alpha_C = \tan^{-1}\frac{0.114}{0.541} = 11.9°.$$
$$x_m = 0.975 - 0.105 = 0.87.$$

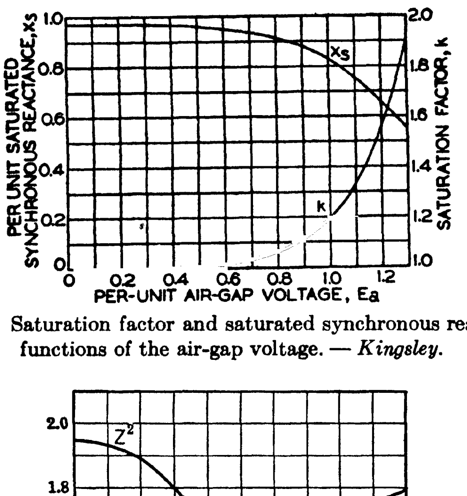

FIG. 108. Saturation factor and saturated synchronous reactance as functions of the air-gap voltage. — *Kingsley.*

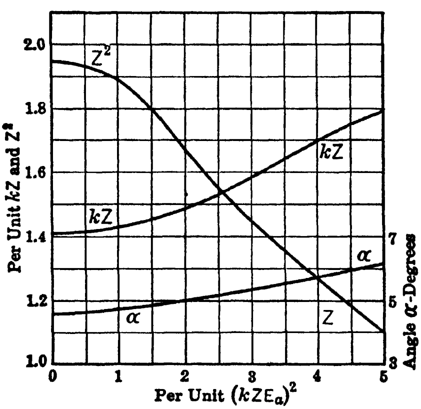

FIG. 109. Auxiliary curves for example. — *Kingsley.*

From the values of α, kz and z^2 calculated above, and similar calculations for other assumed values of E_a, the auxiliary curves of figure 109 are plotted. Equation (384) becomes, on substitution of known values,

$$(kzE_a)^2 = 1.54 + 1.55 \cos (\delta - 11.9°).$$

Corresponding to assumed values of δ in this equation, values of $(kzE_a)^2$ may be determined, and entering figure 109, the values of α, kz and z^2 for substitution, along with the appropriate δ, in (382) and (383). Thus for $\delta = 75°$,

$$\begin{aligned}(kzE_a)^2 &+ 1.54 + 1.55 \cos 63.1° \\ &= 2.24. \\ \alpha &= 5.1°. \\ kz &= 1.51. \\ z^2 &= 1.62. \\ P_a &= \frac{1.61 \times 1.00}{1.51} \sin 69.1° + \frac{1.61^2 \times 0.114}{1.62} \\ &= 1.13. \\ P_R &= \frac{1.61 \times 1.00}{1.51} \sin 80.1° - \frac{1.00^2 \times 0.114}{1.62} \\ &= 0.97.\end{aligned}$$

These are two of the points on the curves of figure 110.

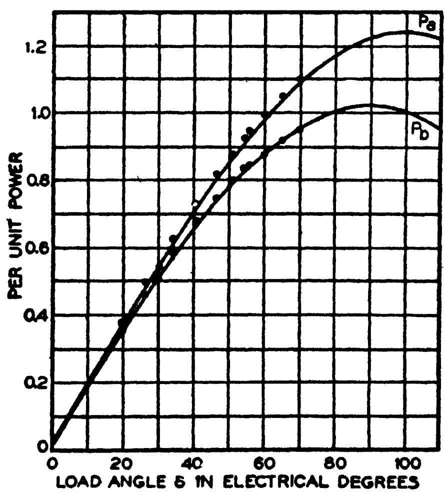

FIG. 110. Calculated power-angle curves of the system of the example. Test points are shown by small circles. — *Kingsley.*

The maximum air-gap power may be computed directly, without plotting the entire power-angle curve, subject only to a small error in α, which depends on the degree of saturation. It is seen from (383) that P_R is a maximum when $\delta - \alpha = 90°$. In the sample computation for $\delta = 75°$, it was found that $\alpha = 5.1°$. Assuming that α will be 5.0° at pull-out, it follows that δ will be 95°.

$$(kzE_a)^2 = 1.73,$$
$$\alpha = 4.95°,$$
$$kz = 1.47,$$
$$z^2 = 1.73.$$

Therefore the maximum value of

$$P_a = \frac{1.61 \times 1}{1.47} + \frac{1.61^2 \times 0.114}{1.47^2}$$
$$= 1.23.$$

The corresponding value of P_R, for the same angle δ, is 1.01, but the maximum value of P_R occurs at an angle of $\delta = 85°$ approximately, and is 1.02.

Salient-pole machines. If the machines have salient poles, as is usual with hydroelectric generators, synchronous reactors and synchronous motors, then in order to determine the machine characteristics with

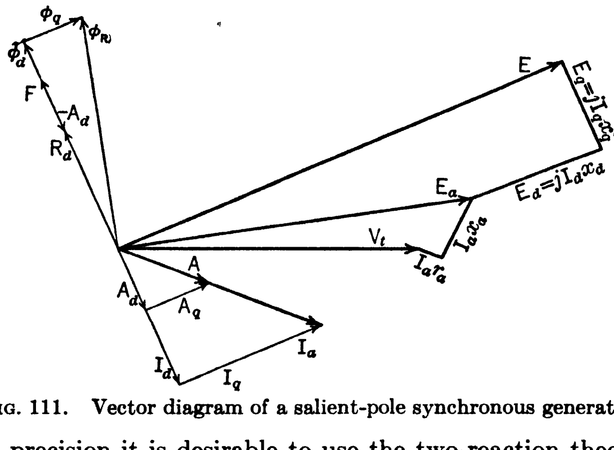

FIG. 111. Vector diagram of a salient-pole synchronous generator.

sufficient precision it is desirable to use the two-reaction theory. The field magnetomotive force F acts along the direct field axis, but the armature magnetomotive force has, in general, a line of action forming some angle with the pole structure. Refer to the vector diagram of figure 111. It is convenient to consider this magnetomotive force divided into two components, one acting along the direct pole axis, and the other along an axis half way between adjacent poles, called the quadrature axis. The flux due to the total direct-axis magnetomotive force is determined from the no-load saturation curve, and the flux due to the quadrature-axis magnetomotive force is added vectorially to this to obtain the total flux. As shown in the vector diagram,

TABLE XXII *

AVERAGE PER-UNIT MACHINE PARAMETERS

	Turbo-alternators	Water-Wheel Generators	Synchronous Motors	Synchronous Condensers	Induction Motors
Synchronous reactance — direct axis	1.10	1.00	1.00	1.50	3.00
Synchronous reactance — quadrature axis	1.10	0.65	0.60	0.85	
Transient reactance — direct axis	0.18	0.30	0.30	0 40	0.15
Transient reactance — quadrature axis	0.18	0.65	0.60	0.85	
Negative phase sequence reactance	0.14	0.40	0.22	0.28	0.15
Inertia constant	16.0	6.0	4.5	3.0	1.0

* Park and Bancker.

reactances equivalent to each of the components of armature reaction are used. Table XXII shows average values of reactance and inertia constants for common types of machines.

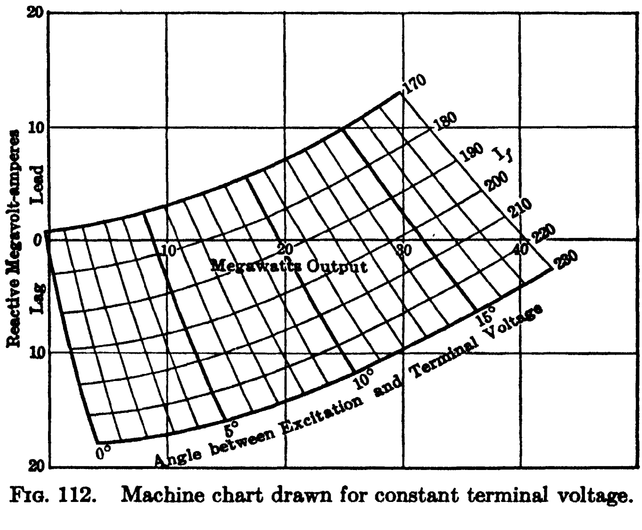

FIG. 112. Machine chart drawn for constant terminal voltage.

Machine charts. Whether the machines have cylindrical rotors or salient poles, it is usually necessary, in a circuit at all complicated, to prepare charts showing the characteristics over the entire range which will be covered, so that trial methods may be used with economy of time. A machine chart, drawn for constant terminal voltage, is shown in figure 112.

86. Transient Stability. Loss of synchronism may be caused by transient conditions due to such events as sudden large increments of load, short circuits and switching operations to clear the faults, and in fact any shock to the system. In connecting a large generator to the line, it is important to use all possible care to do it with minimum shock, which means that the speed should be synchronous, the acceleration zero, the terminal voltage the same as the system voltage at the synchronizing point and the phase relations of voltage the same. In disconnecting it, the active and reactive power should be made zero, as nearly as possible, before the switch is opened.

If a given link in a system has a steady-state power limit of, say, 100,000 kw, and is carrying 60,000 kw, a *sudden* addition to the load of 30,000 kw would probably cause loss of synchronism, just as in the mechanical analog the arms would, through inertia effects, swing beyond 90 degrees of separation, and then fall apart.

The swings of synchronous machines are not so simple as those of loaded linear springs, because the power-angle or torque-angle characteristics of the machines are not linear. For swings of considerable magnitude, such as are contemplated in transient stability studies, there will be variations in field current, regulator action, governor and prime-mover action, as well as gain and loss of stored energy of inertia as the rotors gain and lose speed in their swings, and changes in the main electric circuit relations.

If a correct power-angle or torque-angle curve is plotted, then the effect of a sudden addition of load in causing a swing may be determined by using the method illustrated in figure 113. Let the machine whose power-angle curve is shown be operating at point P_0 in the steady state. A sudden increment of load is imposed, of sufficient magnitude to bring the new steady-state operating condition to point P_2, provided there is no loss of synchronism. The new load will cause the machine to slow down, and fall back in phase relative to the rest of the system. There is a deceleration as long as the operating point lies below P_2, and so when point P_2 is reached the rotor is rotating at less than synchronous speed. The load angle δ will therefore continue to increase beyond this point. The torque developed in the machine is now greater than the load torque, and the rotor accelerates. Even though it is accelerating, the load angle δ will continue to increase until synchronous speed has been reached again. The amount of overshoot which will occur is determined by the amount of kinetic energy required to accelerate the rotor from its minimum speed, which occurs in passing through P_2, back to the speed synchronous with the rest of the system. The swing will reach point P_1, corresponding to load angle δ_1, at its maximum, this point being determined from the fact that the area A_2 is to be made equal to A_1.

The proof of this statement can be shown most simply by noting first that accelerations due to given unbalanced forces are the same if the system is initially at rest, as they are if it is moving at synchronous speed. Reasoning on the basis of a system or rotor initially at rest, the torque deficiency of the machine during the time the operating point is traveling from P_0 to P_2 acts to give the rotor a negative acceleration, and the work done is the area A_1 or the integral of the torque deficiency

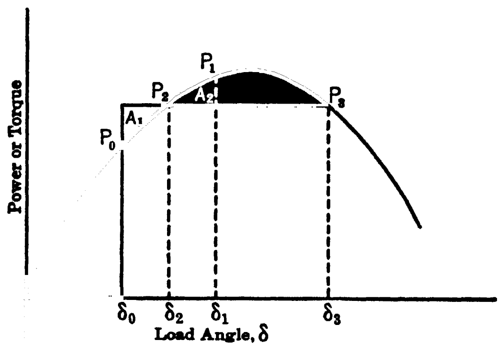

FIG. 113. Equal-area criterion for transient stability.

with respect to angle δ. To get the rotor back to its original (hypothetical) stationary condition, the same amount of work A_2 is required, of the opposite sign.

It is important to note that the maximum transient swing for stable operation is not limited to the peak of the power-angle curve. If the initial operating point P_0 had been considerably lower, with a larger increment so that P_2 still remains the new steady-state point, the swing might reach as far as point P_3 without loss of synchronism. It may be observed that acceleration will be taking place, owing to an excess of generated torque, throughout the whole region between P_2 and P_3, and not merely in that part of the region to the left of the peak of the curve.

When there are two or more stations in the system undergoing rotor acceleration, it is the relative accelerations rather than the absolute values which are the criterion of the pull-out point. The energy converted from kinetic mechanical form to electrical depends naturally on the absolute change in velocity. The changes in velocities of two machines having equivalent inertia moments M_1 and M_2 will be inversely as their moments for a given condition of power unbalance. The relative acceleration will be proportional to

$$\frac{T}{M_1} + \frac{T}{M_2} = \frac{T}{M_1M_2/(M_1 + M_2)},$$

and so the actual system may be replaced by one in which one machine has a moment of inertia $\frac{M_1M_2}{M_1 + M_2}$ and the other an infinite moment. This substitution is permissible only when the main power flow is uninterrupted from end to end of the link, such as when a load is changed or one of two parallel lines cut out. If there is a short circuit near one machine, it may draw much more power from that machine and cause decelerations not even approaching the ratio given above.

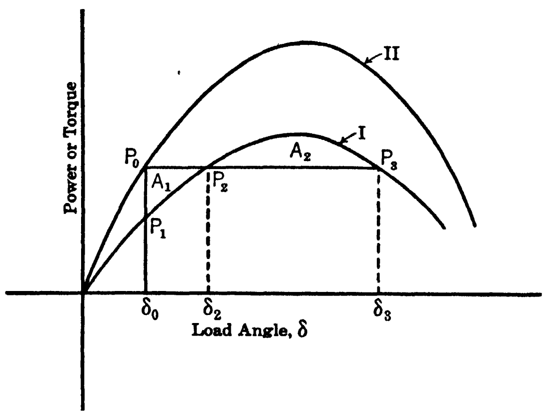

FIG. 114. Effect of opening a parallel line.

The use of power-angle curves in determining the amplitude of swing when one of two or more parallel lines connecting two stations is opened is illustrated in figure 114. Curve II is the power-angle curve for the original system, which is operating in the steady state at point P_0. One of the parallel lines is now opened, leaving a system with power-angle characteristic as given by curve I. The power angle is still δ_0, however, and a swing will develop just as though the final system alone had been operating at P_1, and had been given a sudden increment of load sufficient to bring it to point P_2 for steady-state operation. If area A_2 is larger than A_1, the two stations will remain in synchronism.

A single power-angle curve for a system does not specify its transient behavior precisely for all possible different types of shocks. Step-by-step analyses may be used to take into account such factors as changes in terminal voltage and excitation, action of damper windings, regulator and governor and prime-mover characteristics.

If the system is complicated, it may be simplified by reduction to a two-machine or a three-machine system as nearly as possible equivalent, or the system may be duplicated in miniature on an alternating-current calculating table, and readings on instruments made for steady-state solutions corresponding to known relations of voltage, power and angle. These can serve in the step-by-step solution to determine accelerations, velocities, etc., for computing the conditions for the next step. The period of angular swings in large systems is of the order of 1 or 2 seconds — so long that it may be assumed without serious error that the alternating-current circuit relations are equal at each moment to the steady-state relations corresponding to the existing load angles and excitations. The calculating board then serves to solve this quasi-steady-state electric circuit problem, and to yield results which will indicate the accelerations, velocities and rates of change used in the step-by-step solution for machine swings and field current changes.

87. Methods of Improving System Stability. The steady-state power limit between two stations is increased by reducing the impedance between them, whether this be accomplished by adding parallel lines or machines, or by using machines of lower inherent impedance. Higher excitation voltages also increase the power limit, as does the use of a quick-response excitation system.

Transient stability is improved by any reduction in the number and severity of system faults. Thorough protection against lightning is thus an important factor. Quickly acting circuit breakers for removing faults from the system are advantageous, and automatic reclosing breakers give further help. High-speed excitation helps to maintain synchronism during a fault by quickly increasing the excitation voltage. The type of neutral grounding used has considerable bearing on the severity of most types of short circuit. High neutral grounding impedance limits the fault current to ground but has the disadvantage of requiring more insulation in the transformer windings and on the lines. High-speed governors help by quickly adjusting generator input to load. The use of damper windings in the pole faces has conflicting effects, but produces a net improvement in stability. The use of machines of higher inertia improves stability, other things being equal.

The extent to which these various artifices should be applied is a function of their costs. An excellent discussion of this subject, with curves of relative costs, may be found in the A.I.E.E. committee report on stability in Electrical Engineering for February, 1937.

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PROBLEMS ON CHAPTER VIII

Prob. 1–8. Calculate the steady-state stability limit imposed by the line alone for a 60-cps single-circuit three-phase line operating with 220 kv at both ends. The conductors are 800,000-cir-mil standard copper cable. Make the calculation for lengths of 100, 200, 300, 400 and 500 miles, and plot a curve between length and power limit. Spacing is 25 ft equilateral.

Prob. 2–8. Assume transformers at both ends of the lines of problem 1–8, having reactances of 10 per cent, based on their capacities, which are to be assumed 80 per cent of the steady-state power limit of the line and transformers combined. Calculate and plot the power limits for lengths of 100, 200, 300, 400 and 500 miles.

Prob. 3–8. Calculate and plot the power-angle curves for the cylindrical-rotor machine whose characteristics are shown in figure 106, when it is feeding an infinite bus through impedances as stated in the example. The bus voltage V is 1.00 and E_f is 1.50 on a per-unit basis. What is the maximum air-gap power?

Prob. 4–8. Using the data from figure 110, calculate and plot a curve between initial receiver air-gap power and maximum permissible sudden increment, using the equal-area criterion.

Prob. 5–8. What is the maximum power which can be carried without loss of stability through the transient originated by opening a parallel line, if the power-angle curves are those of figure 114? Express in terms of the peak of curve I, for the line which remains in service.

Prob. 6–8. Prove that the mechanical analog shown in figure 103 will become unstable, for arms of equal length, when the arms are 90 degrees apart.

CHAPTER IX

MECHANICAL PRINCIPLES

88. Line Construction. Mechanical factors of safety to be used in transmission line design should depend to some extent on the importance of continuity of operation in the line under consideration. In general, the strength should be such as to provide against the worst *probable* weather conditions.

The supporting structures in most of the important transmission links in large systems are steel towers. Wood poles and wood frames are used satisfactorily in many lines for voltages as high as 110,000. The steel towers are more expensive, but with proper care are more durable, as well as stronger and practically immune to damage from lightning and fire. Tower design is a specialized art which has had a development empirical in considerable part, owing to the fact that towers are usually indeterminate structures. The designs are ordinarily worked out and tests made by the manufacturers of towers. These designs are subject to further checking and test by the users, if it is considered desirable.

Wood poles last 10 to 12 years on the average, as against two to three times that long for steel. The insulating value of the wood seems to be of some help in reducing the number of flashovers, but when a flashover does occur on wood it is likely to splinter it. This may be guarded against to some extent by running a ground wire down each pole, but then the insulating value is lost.

An interesting collection of opinion and experience in regard to the relative merits of wood and steel may be found in one of the references at the end of the chapter.

Tower height is dependent on the length of span. With long spans, relatively few towers are required, but they must be tall and correspondingly costly. It is not usually feasible to determine the tower height and span length on the basis of direct construction costs, because lightning hazards increase greatly as the height of the conductors above ground is increased. For this reason, also, horizontal spacing is favored, in spite of the wider right of way required.

The conductor clearance to ground at the time of greatest sag should not be less than some specified distance, usually between 20 and 40 feet, depending on the voltage, on the nature of the country and on local ordinances or state laws. The greatest sag may occur on the hottest day of summer, on account of the expansion of the wire with the heat; or it may occur in the winter, in spite of the shrinkage due to the cold, owing to the formation of a heavy coating of ice on the wires. Many power companies in the northern part of the United States and in Canada have special provisions for melting ice from their power lines by forcing heavy currents through them. A description of some of the methods used may be found in one of the references.

On some lines, every tower is made a strain tower, capable of standing even if all the wires on one side are broken. On other lines, several flexible or semi-flexible towers are used between strain towers, in the interest of economy.

Stretching the conductors as tightly as is consistent with safety aids in keeping down the height of tower required.

89. Conductors. The principal types of conductor in general use are:

1. Solid copper wires, used in sizes up to no. 0000, having diameter 0.460 inch.

2. Stranded copper cable, which in large sizes is preferable to solid copper because it is more flexible and hence easier to ship and handle, and is less subject to crystallization. Also, it is stronger.

3. Hollow copper cable. This type affords a large diameter when the cross section of copper required for best economy is small. Corona losses may thereby be reduced or eliminated. Skin effect and inductance are somewhat less for the hollow design as compared with standard concentric stranded cable.

4. Steel-reenforced aluminum cable (A.C.S.R.). This type of conductor has steel strands at its center, surrounded by aluminum strands. It is larger and lighter than a standard stranded copper cable of the same resistance. It shares with the hollow copper conductor the disadvantage inherent in large cables of collecting a greater weight of ice. Aluminum is in general somewhat more vulnerable to vibration troubles than copper is, and vibration dampers are used to a greater extent on aluminum cables.

5. Copper-clad steel wire is made by pouring molten copper around a billet of steel, and afterward rolling and drawing it down to the desired size. It is used as solid wire and as cable, sometimes in combination with plain copper strands.

6. Copper and bronze combination cables. Bronze strands are sometimes used with copper to give additional strength and at the same time to contribute more toward the conductance than steel strands would do.

Solid aluminum wire is not used, for mechanical reasons.

In small sizes, a three-strand twisted cable is sometimes favored because it is free from vibration troubles. Vibration is usually caused

TABLE XXIII

Material	Standard Annealed Copper	Hard-Drawn Copper	Aluminum	Steel	Bronze
Relative conductivity	1.00	0.97	0.61	0.09 to 0.16	0.15 to 0.35
Specific gravity	8.89	8.89	2.70	7.78	8.54 to 8.89
Thermal coefficient of linear expansion per deg. at 20 C	16.5×10^{-6}	16.5×10^{-6}	23×10^{-6}	12.6×10^{-6}	$15 \text{ to } 17 \times 10^{-6}$
Density, pounds per cubic inch	0.321	0.321	0.0975	0.281	0.308 to 0.321
Elastic limit, pounds per square inch	....	25,000 to 30,000	25,000 to 30,000	40,000 to 150,000	25,000 to 65,000
Breaking strength, pounds per square inch	34,000	50,000 to 65,000	23,000 to 30,000	70,000 to 250,000	60,000 to 140,000
Modulus of elasticity, pounds per square inch	....	$16 \text{ to } 18 \times 10^{6}$	10,000,000	$24 \text{ to } 30 \times 10^{6}$	$15 \text{ to } 18 \times 10^{6}$

by the regular formation of eddies on the lee side of wires, recurrently above and below the level of the center. Three-strand cables have a surface configuration so deeply indented that the eddies are broken up and do not cause the usual trouble.

The principal physical characteristics of conductor materials which affect their mechanical performance are given in table XXIII.

Ice loading may be taken as weighing 0.033 pound per cubic inch.

Many of the figures in table XXIII are only average or approximate values, and detail test results should be used where possible. For the same material, smaller wires show greater strength owing to the additional working to which they have been subjected. Stranded conductors have constants different from those of solid wires, and manufacturers' test results should be used.

90. Sag and Stress Analysis. Sags and stresses in conductors are dependent on the initial tension put on them when they are clamped in place, and are due to the dead weight of the conductors themselves, to ice and sleet that may adhere to them, and to wind pressure. Sags and stresses vary with temperature on account of the thermal expansion and contraction of the conductors. In case one or more conductors break, extra stresses, both transient and steady, are set up in neighboring parts of the transmission structure.

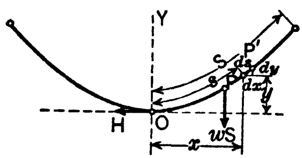

FIG. 115. Conductor suspended between supports at the same elevation.

Figure 115 represents a span of wire with the two supports at the same elevation, and separated by a horizontal distance $2\,l$. The following symbols will be used:

v = weight or vertical force per unit length of conductor.
h = horizontal force per unit length due to wind pressure.
$w = \sqrt{v^2 + h^2}$ = resultant force per unit length.
H = horizontal component of tension in conductor.
T = tension in conductor at point x.
x = abscissa, horizontal distance from center of span to any arbitrary point on the curve.
y = ordinate of any point on the curve.
d = maximum ordinate = sag.
θ = angle curve makes with horizontal line in its plane, at any point x, y.
s = length of curve from lowest point 0, 0 to x, y.

There is no y component of tension at 0, 0; hence the y component at x, y must be equal to the loading between 0, 0 and x, y, that is, to ws.

We have, therefore:

$$\frac{dy}{dx} = \tan\theta = \frac{ws}{H}$$

$$= \frac{w}{H}\int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx. \tag{385}$$

Differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \frac{w}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{386}$$

Putting

$$\frac{dy}{dx} = p,$$

$$\frac{dp}{dx} = \frac{w}{H}\sqrt{1 + p^2}. \tag{387}$$

$$\frac{w}{H}dx = \frac{dp}{\sqrt{1 + p^2}}. \tag{388}$$

Integrating,

$$\frac{wx}{H} = \sinh^{-1} p + C_1, \tag{389}$$

where C_1 is a constant of integration. At the lowest point of the curve, $x = 0$ and $p = 0$, so it follows that in equation (389) $C_1 = 0$, and we have

$$p = \sinh\frac{wx}{H}. \tag{390}$$

$$\therefore\ dy = \sinh\frac{wx}{H}dx, \tag{391}$$

$$y = \frac{H}{w}\cosh\frac{wx}{H} + C_2. \tag{392}$$

When $x = 0$, $y = 0$ if we take the origin at the lowest point, so

$$y = \frac{H}{w}\left(\cosh\frac{wx}{H} - 1\right) \tag{393}$$

is the equation of the curve. It is called a catenary.

The total tension at any point x is

$$T = H\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= H\cosh\frac{wx}{H}. \tag{394}$$

The sag for a span of length $2l$ between supports on the same level is:

$$d = \frac{H}{w}\left(\cosh\frac{wl}{H} - 1\right)$$

$$= l\left[\frac{1}{2}\left(\frac{wl}{H}\right) + \frac{1}{24}\left(\frac{wl}{H}\right)^3 + \frac{1}{720}\left(\frac{wl}{H}\right)^5 \cdots\right]. \tag{395}$$

The distances y and d are measured in the direction of the resultant loading w; they are vertical if there is no wind pressure.

91. Supports at Different Elevations. Consider a span between two supports whose elevations differ by a distance q, the horizontal spacing being $2l$ as before. The wind pressure is assumed zero in this first analysis. Refer to figure 116. Let the horizontal distance from the lowest point of the catenary to the lower support be x_1, and the corresponding vertical distance y_1. Then we have

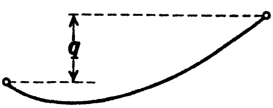

FIG. 116. Conductor with supports at different levels.

$$y_1 = \frac{H}{w}\left(\cosh\frac{wx_1}{H} - 1\right), \tag{396}$$

$$y_1 + q = \frac{H}{w}\left[\cosh\frac{w(2l - x_1)}{H} - 1\right]. \tag{397}$$

Subtracting,

$$q = \frac{H}{w}\left[\cosh\frac{w(2l - x_1)}{H} - \cosh\frac{wx_1}{H}\right]$$

$$= \frac{2H}{w}\sinh\frac{wl}{H}\sinh\frac{w(l - x_1)}{H}. \tag{398}$$

$$\therefore x_1 = l - \frac{H}{w}\sinh^{-1}\frac{qw}{2H\sinh\frac{wl}{H}}. \tag{399}$$

To check this, put $q = 0$. Then $x_1 = l$, as before. In some cases x_1 may be negative; there will then be no horizontal point in the span. This might easily happen if the line runs up a steep mountainside.

After the low point has been determined, each side may be treated independently by means of the formulas already developed for the span with supports at the same elevation.

92. Development of Working Formulas. Supports at Equal Elevation. The formulas developed thus far involve the horizontal component of tension H, but it is more convenient to express them in terms of total tension. The formulas which follow have been derived by H. B. Dwight (see reference at end of chapter).

We have from equation (394),

$$T = H\cosh\frac{wx}{H} = H\left[1 + \frac{1}{2}\left(\frac{wx}{H}\right)^2 + \frac{1}{24}\left(\frac{wx}{H}\right)^4 + \cdots\right]. \tag{400}$$

Dividing both sides into wx gives

$$\frac{wx}{T} = \frac{wx}{H} - \frac{1}{2}\left(\frac{wx}{H}\right)^3 + \frac{5}{24}\left(\frac{wx}{H}\right)^5 \cdots. \tag{401}$$

If we neglect terms of higher order than the fifth, we may write from (401), raising both sides to the fifth power,

$$\left(\frac{wx}{T}\right)^5 = \left(\frac{wx}{H}\right)^5. \tag{402}$$

Cubing both sides of (401) and discarding terms of orders higher than the fifth gives

$$\left(\frac{wx}{T}\right)^3 = \left(\frac{wx}{H}\right)^3 - \frac{3}{2}\left(\frac{wx}{H}\right)^5. \tag{403}$$

Therefore

$$\left(\frac{wx}{H}\right)^3 = \left(\frac{wx}{T}\right)^3 + \frac{3}{2}\left(\frac{wx}{H}\right)^5. \tag{404}$$

Similarly,

$$\frac{wx}{H} = \frac{wx}{T} + \frac{1}{2}\left(\frac{wx}{T}\right)^3 + \frac{13}{24}\left(\frac{wx}{T}\right)^5. \tag{405}$$

Now substituting (402), (404) and (405) back into (393),

$$y = \frac{H}{w}\left(\cosh\frac{wx}{H} - 1\right) = \frac{H}{w}\left[\frac{1}{2}\left(\frac{wx}{H}\right)^2 + \frac{1}{24}\left(\frac{wx}{H}\right)^4 + \frac{1}{720}\left(\frac{wx}{H}\right)^6 + \cdots\right]$$

$$= x\left[\frac{1}{2}\frac{wx}{T} + \frac{7}{24}\left(\frac{wx}{T}\right)^3 + \frac{241}{720}\left(\frac{wx}{T}\right)^5 + \cdots\right]. \tag{406}$$

$$d = l\left[\frac{1}{2}\frac{wl}{T_m} + \frac{7}{24}\left(\frac{wl}{T_m}\right)^3 + \frac{241}{720}\left(\frac{wl}{T_m}\right)^5 + \cdots\right], \tag{407}$$

in which T_m is the tension at the supports.

By similar methods the following formulas are developed:

$$T_m = wl\left[\frac{1}{2}\frac{l}{d} + \frac{7}{6}\frac{d}{l} - \frac{4}{45}\frac{d^3}{l^3} \cdots\right]. \tag{408}$$

$$L = 2l\left[1 + \frac{2}{3}\frac{d^2}{l^2} - \frac{14}{45}\frac{d^4}{l^4} + \frac{278}{945}\frac{d^6}{l^6}\cdots\right] \tag{409}$$

$$= 2l\left[1 + \frac{1}{6}\left(\frac{wl}{H}\right)^2 + \frac{1}{120}\left(\frac{wl}{H}\right)^4\cdots\right] \tag{410}$$

$$= 2l\left[1 + \frac{1}{6}\left(\frac{wl}{T_m}\right)^2 + \frac{7}{40}\left(\frac{wl}{T_m}\right)^4 + \frac{241}{1008}\left(\frac{wl}{T_m}\right)^6\cdots\right]. \tag{411}$$

$$L_u = L - \frac{wl^3}{AEd}\left[1 + \frac{5}{3}\frac{d^2}{l^2} + \frac{4}{9}\frac{d^4}{l^4} \cdots\right] \tag{412}$$

$$= L - \frac{2\,Hl}{AE}\left[1 + \frac{1}{3}\left(\frac{wl}{H}\right)^2 + \frac{1}{15}\left(\frac{wl}{H}\right)^4 \cdots\right] \tag{413}$$

$$= L - \frac{2\,T_m l}{AE}\left[1 - \frac{1}{6}\left(\frac{wl}{T_m}\right)^2 - \frac{7}{120}\left(\frac{wl}{T_m}\right)^4 \cdots\right]. \tag{414}$$

$$H = T_m\left[1 - \frac{1}{2}\left(\frac{wl}{T_m}\right)^2 - \frac{7}{24}\left(\frac{wl}{T_m}\right)^4 \cdots\right]. \tag{415}$$

In the above equations, L is the total length along the curve, and L_u is the unstressed length, or length minus the amount of stretch. Where L_u is known for one temperature t_1, the value of L_u for a new temperature t_2 is

$$L_{u2} = L_{u1}[1 + \alpha\,(t_2 - t_1)], \tag{416}$$

where α is the coefficient of thermal expansion. A is the area of section and E the modulus of elasticity.

In planning the sag, stress and clearance to ground of a given span, a maximum stress is selected and it is aimed to have this stress develop at the worst probable weather conditions for stress; that is, minimum expected temperature, maximum ice loading and maximum wind. The sag and clearance should be planned so that minimum desired clearance will occur either at maximum summer temperature (augmented by heat losses in conductor) with no wind, or else with maximum ice loading and a temperature of 0 degrees C, the maximum temperature compatible with an ice coating, and again no wind. Wind loading increases the sag in the direction of the resultant loading, but decreases the vertical component. Hence in clearance calculations the effect of wind should not be included, unless horizontal clearance is of importance.

Example. A 2000-ft span over a river is to have its two supports at the same elevation. The conductor weighs 0.783 lb per ft, and the maximum expected ice loading is 1.5 lb per ft. The maximum expected wind pressure (with ice) is 1.1 lb per ft. The allowable tension is 16,000 lb. The area is 617,000 cir mils, and the effective modulus of elasticity is 15,000,000 lb per sq in. The temperature range of the conductor is 120 F to −20 F. The coefficient of thermal expansion is 9.6×10^{-6} per degree F.

Find the minimum required height of the supports to afford a 50-ft clearance above mean water level under all conditions. Find the proper stringing tension at 70 F.

Solution. $T_m = 16{,}000$ at greatest ice and wind loads and at temperature -20 F.

$v = 0.783 + 1.5 = 2.283$. $h = 1.1$. $w = 2.534$ lb per ft. From equations (411) and (414),

$$L_u = 2008.590 - 4.384 = 2004.206 \text{ ft at } -20 \text{ F.}$$
$$\text{At } 32 \text{ F, } L_u = 2004.206(1 + 0.0000096 \times 52) = 2005.206 \text{ ft.}$$
$$\text{At } 70 \text{ F, } L_u = 2004.206(1 + 0.0000096 \times 90) = 2005.938 \text{ ft.}$$
$$\text{At } 120 \text{ F, } L_u = 2004.206(1 + 0.0000096 \times 140) = 2006.900 \text{ ft.}$$

To find the sag at 32 F with ice and no wind, which may be the greatest sag, a successive approximation method will be used. The loading is $w = v = 2.283$ lb per ft.

Try $d = 50$ ft. From (409) and (412), $L_u = 1997.022$. Actually at this temperature L_u must be 2005.206, so we must try a larger sag.

$$\text{Try } d = 100 \text{ ft.} \qquad L_u = 2010.078 \text{ ft.}$$
$$\text{Try } d = 80 \text{ ft.} \qquad L_u = 2004.540 \text{ ft.}$$
$$\text{Try } d = 82.4 \text{ ft.} \qquad L_u = 2005.170 \text{ ft.}$$
$$d = 82.54 \text{ by extrapolation from the last two trials.}$$

Sag may be greatest, however, at 120 F with no wind, and naturally no ice. $w = v = 0.783$ lb per ft.

Try $d = 80$ ft. L_u comes out to be 2007.147, and since we know that the actual value of L_u at 120 F is 2006.900, it is apparent that the sag must be less than 80 ft. Hence the maximum sag is 82.54 ft, and the support height should be 132.5 ft above mean water level.

Now to find T_m at 70 F, with no wind or ice, we have again

$$w = v = 0.783, \quad \text{and} \quad L_u = 2005.938 \text{ ft.}$$

A successive approximation method will be used again, with equations (411) and (414).

$$\text{Try } T_m = 10{,}000 \text{ lb.} \qquad L_u = 2000.584 \text{ ft.}$$
$$\text{Try } T_m = 8{,}000 \text{ lb.} \qquad L_u = 2001.028 \text{ ft.}$$
$$\text{Try } T_m = 6{,}000 \text{ lb.} \qquad L_u = 2004.132 \text{ ft.}$$
$$\text{Try } T_m = 5{,}500 \text{ lb.} \qquad L_u = 2005.511 \text{ ft.}$$

Extrapolating, $T_m = 5340$ lb, the proper tension for stringing at 70 F.

93. Working Formulas. Supports at Unequal Elevations. (Dwight's method.) Consider now again the case of supports at unequal elevations. The horizontal separation of the supports is $2\,l$, and the vertical separation is p. The vertical and horizontal components of loading per unit length are v and h, and their resultant w. Also, let

$$q = \frac{pw}{v}, \tag{417}$$

$$b = \frac{q}{2\,k}, \tag{418}$$

$$\begin{aligned} 2\,k &= \sqrt{4\,l^2 - q^2 + p^2} \\ &= 2\,l - \frac{q^2 - p^2}{4\,l} - \frac{(q^2 - p^2)^2}{64\,l^3} \cdots. \end{aligned} \tag{419}$$

The physical significance of each of these symbols may be made clearer by reference to figure 117, which shows a model span between supports at different elevations. The effect of wind pressure is simulated by setting a rectangular flat sheet of stiff material at an angle, and

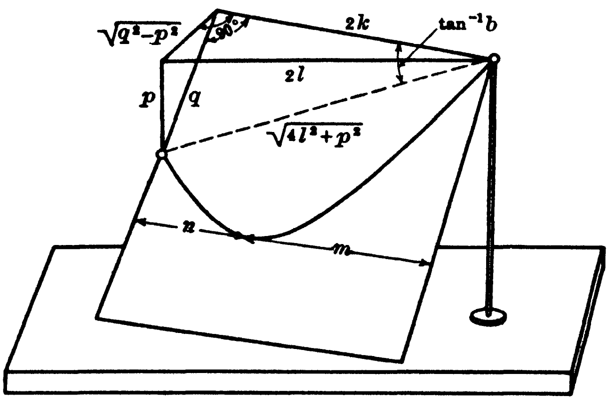

FIG. 117. Model of span having supports at unequal elevations, with the conductor acted on by a resultant force containing a horizontal component and a vertical component.

with both supports in the plane of the sheet. The flexible span is constrained to lie against this sheet. All resultant forces w on the elements of the span act in a direction parallel to the sloping edges of the sheet.

Greatest stress in the span will be produced when the wind force is not acting at right angles to the line of the span marked $2\,l$, but when it is at right angles to the line marked $2\,k$.

By analogy to equation (399), the expression for the distance n in figure 117 may be written

$$n = k - \frac{H}{w}\sinh^{-1}\frac{qw}{2\,H\sinh\dfrac{wk}{H}} \tag{420}$$

$$= k - \frac{H}{w}\sinh^{-1}\frac{b}{1+\dfrac{1}{6}\left(\dfrac{wk}{H}\right)^2+\dfrac{1}{120}\left(\dfrac{wk}{H}\right)^4+\cdots} \tag{421}$$

$$= k - \frac{H}{w}\sinh^{-1} b\left[1-\frac{1}{6}\left(\frac{wk}{H}\right)^2+\frac{7}{360}\left(\frac{wk}{H}\right)^4\cdots\right]. \tag{422}$$

Making use of the inverse hyperbolic sine expansion (convergent for $x^2 < 1$)

$$\sinh^{-1} x = x - \frac{1}{2{\cdot}3}x^3 + \frac{1{\cdot}3}{2{\cdot}4{\cdot}5}x^5 - \frac{1{\cdot}3{\cdot}5}{2{\cdot}4{\cdot}6{\cdot}7}x^7\cdots, \tag{423}$$

expression (422) may be written

$$\begin{aligned} n = k - \Bigg\{&\frac{H}{w}\left(b-\frac{1}{6}b^3+\frac{3}{40}b^5-\frac{5}{112}b^7\cdots\right)\\ &-\frac{k}{6}\left(\frac{wk}{H}\right)\left(b-\frac{1}{2}b^3+\frac{3}{8}b^5\cdots\right)\\ &+\frac{k}{30}\left(\frac{wk}{H}\right)^3\left(\frac{7}{12}b-\frac{17}{24}b^3\cdots\right)\Bigg\}. \end{aligned} \tag{424}$$

The distance m is equal to $2\,k - n$, and so is equal to k plus the expression in the brace in (424).

Converting the expression to one involving T_m instead of H, by the method already illustrated, gives for m,

$$\begin{aligned} m = k\Bigg\{&\left(1-b^2+\frac{2}{3}b^4-\frac{8}{15}b^6\cdots\right)\\ &+\left(\frac{wk}{T_m}\right)^{-1}\left(b-\frac{2}{3}b^3+\frac{8}{15}b^5-\frac{16}{35}b^7\cdots\right)\\ &-\left(\frac{wk}{T_m}\right)\left(\frac{2}{3}b+0+\frac{2}{45}b^5\cdots\right)\\ &-\left(\frac{wk}{T_m}\right)^2\left(\frac{2}{3}b^2+0\cdots\right)\\ &-\left(\frac{wk}{T_m}\right)^3\left(\frac{16}{45}b+\frac{8}{9}b^3\cdots\right)\Bigg\}. \end{aligned} \tag{425}$$

After m and n have been determined, each side of the span may be handled separately by the same formulas used for the condition of supports at the same elevation.

Example. (This example is from Dwight's paper, *loc. cit.*) A transmission span crossing the Mississippi river has a horizontal length $2\,l = 4279$ ft, and a difference of elevation $p = 185.5$ ft between supports. Under the worst loading conditions, $v = 2.623$ lb per ft and $h = 2.036$ lb per ft. The tension T_m at the higher support is 33,000 lb. The conductor alone weighs 1.684 lb per ft. Find the deflection from the upper support. These conditions hold at 0 deg. with $\frac{1}{2}$ in. of ice.

Solution.

$$w = \sqrt{2.623^2 + 2.036^2} = 3.322 \text{ lb per ft.}$$

$$q = \frac{pw}{v} = 235.0 \text{ ft.}$$

$$2\,k = 4276.57 \text{ ft.}$$

$$b = \frac{q}{2\,k} = 0.05495.$$

By equation (425),

$$\begin{aligned} m &= 2132 + 545 - 17.0 - 0.2 - 0.4 \\ &= 2659 \text{ ft.} \\ n &= 1617.57 \text{ ft.} \end{aligned}$$

By equation (407), using m in place of l there,

$$\begin{aligned} d &= 356 + 15 + 1 \\ &= 372 \text{ ft,} \end{aligned}$$

which is the sag in the plane of the conductor from the upper support. The vertical component of the sag is $372 \times \frac{2.623}{3.322} = 294$ ft.

Example. Find the sag in the Mississippi river span at 120 F, with no wind. Take $E = 30{,}000{,}000$ lb per sq in. and $A = 0.3952$ sq in. The conductor is aluminum cable, steel reenforced, but the calculation is to be made on the basis of all the stress being taken by the steel.

Solution. Under the heavily loaded conditions of the preceding example, the value of unstressed length may be calculated by (409) and (412). The sag from the upper support is 372 ft and from the lower support 137 ft.

$$L_u = 2659\left[1 + \frac{2}{3}\left(\frac{372}{2659}\right)^2 - \frac{14}{45}\left(\frac{372}{2659}\right)^4 + \frac{278}{945}\left(\frac{372}{2659}\right)^6 \cdots\right]$$

$$- \frac{3.322 \times 2659^3}{2 \times 11{,}856{,}000 \times 372}\left[1 + \frac{5}{3}\left(\frac{372}{2659}\right)^2 + \frac{4}{9}\left(\frac{372}{2659}\right)^4 \cdots\right]$$

$$+ 1617.57\left[1 + \frac{2}{3}\left(\frac{137}{1617.57}\right)^2 - \frac{14}{45}\left(\frac{137}{1617.57}\right)^4 + \frac{278}{945}\left(\frac{137}{1617.57}\right)^6 \cdots\right]$$

$$- \frac{3.322 \times 1617.57^3}{2 \times 11{,}856{,}000 \times 137}\left[1 + \frac{5}{3}\left(\frac{137}{1617.57}\right)^2 + \frac{4}{9}\left(\frac{137}{1617.57}\right)^4 \cdots\right]$$

$$= 2693.44 - 7.32 + 1625.30 - 4.38$$

$$= 4307.04 \text{ ft.}$$

$$L_u \text{ at } 120 \text{ F} = 4307.04\,(1 + 120 \times 0.0000070)$$

$$= 4310.66 \text{ ft.}$$

It is now necessary to use a method of successive approximation. Under the specified new conditions, $2k = 2l = 4279$ ft, $q = p = 185.5$ ft and $w = v = 1.684$ lb per ft. A new determination of m and n must be made for each trial.

$$b = \frac{q}{2k} = \frac{185.5}{4279} = 0.04335.$$

Try $H = 18{,}000$ lb. $\dfrac{wk}{H} = 0.2002$. Then from (424),

$$m = 2139.5 + 2139.5\left\{\frac{1}{0.2002}\left(0.04335 - \frac{1}{6}0.04335^3 + \frac{3}{40}0.04335^5 \cdots\right)\right.$$

$$- \frac{1}{6} \times 0.2002\left(0.04335 - \frac{1}{2}0.04335^3 + \frac{3}{8}0.04335^5 \cdots\right)$$

$$\left. + \frac{1}{30} \times 0.2002^3\left(\frac{7}{12}0.04335 - \frac{17}{24}0.04335^3 \cdots\right)\right\}$$

$$= 2139.5 + 463.2 - 3.1 + 0.014$$

$$= 2599.6 \text{ ft.}$$

$$n = 2k - m = 1679.4 \text{ ft.}$$

From equations (410) and (413) the value of L_u corresponding to the assumed value of H may now be worked out. We have $\dfrac{wm}{H} = 0.2433$ and $\dfrac{wn}{H} = 0.1570$.

$$L_u = 2599.6\left[1 + \frac{1}{6}0.2433^2 - \frac{19}{120}0.2433^4 \cdots\right]$$
$$- \frac{18{,}000 \times 2599.6}{11{,}856{,}000}\left[1 + \frac{1}{3}0.2433^2 + \frac{1}{15}0.2433^4 \cdots\right]$$
$$+ 1679.4\left[1 + \frac{1}{6}0.1570^2 - \frac{19}{120}0.1570^4 \cdots\right]$$
$$- \frac{18{,}000 \times 1679.4}{11{,}856{,}000}\left[1 + \frac{1}{3}0.1570^2 + \frac{1}{15}0.1570^4 \cdots\right]$$
$$= 2623.82 - 4.02 + 1686.11 - 2.57$$
$$= 4303.34 \text{ ft.}$$

This value is too small, and so it is necessary to try a value of H less than before.

Try $H = 15{,}000$ lb.

$$\frac{wk}{H} = 0.2402.$$
$$m = 2521.8 \text{ ft.}$$
$$n = 1757.2 \text{ ft.}$$
$$L_u = 2553.00 - 3.28 + 1768.15 - 2.25$$
$$= 4315.62 \text{ ft.}$$

The true value of L_u has been bracketed by the two trials, and by interpolation it is found that $m = 2553$ ft and $H = 16{,}200$ lb. Corresponding to these two values, the sag from the upper support, by (395), is equal to 341 ft.

94. Precision in Sag and Stress Computation. Sags and stresses in transmission spans undergo large variations when there are minute changes in the difference between the span length and the conductor length. In one of the examples it was seen that a difference of less than 6 inches in unstressed length of conductor in a 2000-foot span was sufficient to cause a change in maximum tension from 10,000 to 8,000 pounds.

For this reason, these small differences must be determined with accuracy, and if they are found by taking differences of large numbers, as is usual, these large numbers need to be carried to five or six significant figures. The distances represented may not be known with the degree of precision to justify six significant figures, but any small error of measurement would have relatively little effect. For example, a span of 1000 feet with $L_u = 1001$ feet would have very nearly the same sag and stress as a span of 1000.5 feet with $L_u = 1001.5$.

Using five or six significant figures does not necessitate long multiplication or calculating machine. The slide rule is satisfactory provided the work is properly organized. As an example, consider the computation of L for a span of $2\,l = 1012.00$ feet and a sag of 40 feet. From (409),

$$
\begin{aligned}
L &= 2\,l\left[1 + \frac{2}{3}\frac{d^2}{l^2} - \frac{14}{45}\frac{d^4}{l^4}\cdots\right] \\
&= 1012.00\,[1 + 0.00417 - 0.00001\cdots] \\
&= 1012.00 + 1012 \times 0.00416 \\
&= 1012.00 + 4.21. \\
&= 1016.21 \text{ ft.}
\end{aligned}
$$

Note how the work was performed. All multiplication was done on a 10-inch slide rule, yet the result is found to six significant figures.

An advantage of the convergent series form of solution is that a single method suffices for computation on all spans, long or short, and yet full advantage is taken of the possible simplification in the computation of short spans because only the first one or two terms of the series need be used provided they give sufficient convergence.

95. Temperature-Sag and Temperature-Tension Charts. For use in the field work of stringing the conductors, and adjusting them to the proper sag and tension, it is desirable to have a stringing chart from which the engineer or the foreman can read the values of sag and tension from a knowledge of span length and temperature.

Example. A transmission line over flat country is to have no. 0000 stranded hard-drawn copper conductors, and standard spans of 1000 ft. Assuming maximum loading of $h = 1.29$ and $v = 1.02$ lb per ft to produce maximum tension of 5000 lb at 0 deg F, prepare a stringing chart to cover the temperature range from 40 to 100 F. The conductor alone weighs 0.645 lb per ft, and its modulus of elasticity is 15,000,000 lb per sq in. Its area is 0.1662 sq in.

Solution. Under the conditions of heavy loading,

$$
\begin{aligned}
w &= \sqrt{1.02^2 + 1.29^2} = 1.645 \text{ lb per ft} \\
L_u &= 1000\,[1 + 0.00451 + 0.00013] - 2.006\,[1 - 0.00451 - 0.00004] \\
&= 1004.64 - 2.00 = 1002.64 \text{ ft at 0 deg F.}
\end{aligned}
$$

Let d now equal 38 ft, with no ice or wind.

$$
\begin{aligned}
L_u &= 1000\,[1 + 0.00385 - 0.00001] - 0.850\,[1 + 0.0096] \\
&= 1003.84 - 0.86 = 1002.98 \text{ ft.}
\end{aligned}
$$

This represents an increase of 0.34 ft in L_u over its value at 0 deg F, and with thermal expansion coefficient of 9.2×10^{-6} per deg F, this corresponds to a temperature of 37 F. The tension, by (408), is 2150 lb.

Let $d = 39$ ft. Then $L_u = 1003.21$ ft, temperature is 62 F and tension is 2098 lb.

Let $d = 40$ ft. $L_u = 1003.43$ ft, temperature is 86 F and tension is 2043 lb.

Let $d = 41$ ft. $L_u = 1003.68$ ft, temperature is 113 F and tension is 1999 lb.

The curves are shown in figure 118.

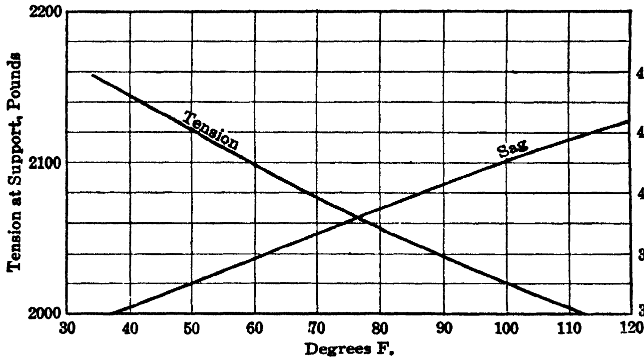

FIG. 118. Stringing chart for a 1000-foot span.

96. Locating the Towers. A convenient way of locating the towers on a profile drawing of the route to be followed is illustrated in figure 119. A template is prepared, on celluloid or tracing cloth, bearing

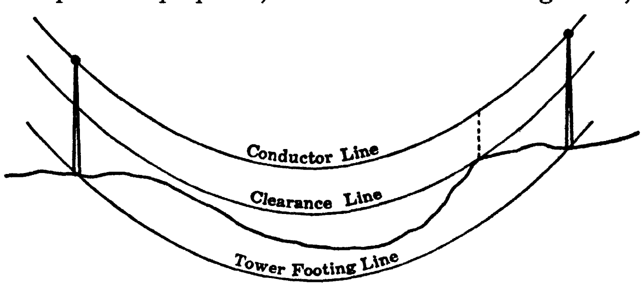

FIG. 119. Use of template in locating tower footings.

three curves, as shown. These curves are catenaries and are drawn to the same horizontal and vertical scales as the profile of the line. The upper curve represents the conductor. The middle curve is below the upper one by a uniform vertical distance equal to the desired minimum vertical clearance to ground. The lower curve is below the upper one by a uniform vertical distance equal to the height of a standard tower, measured to the point of support of the conductor. Assuming the template to be made of celluloid, it may be kept properly oriented by keeping its base against a T-square. If the left tower in figure 119 has already been located, then the location of the right-hand tower is determined by adjusting the template position so that the conductor line passes through the point of support on the left-hand tower, and the clearance line is tangent to ground and nowhere dips below. It may be tangent at only one point, or at two or more.

On some lines, standard towers are selected of a definite height, and then in some locations are used in conjunction with extensions which form the base of the tower. A typical standard tower might be 70 feet high, and 20-foot extensions might be used under 5 or 10 per cent of the towers. When this is done, a fourth line may be drawn on the template to indicate the location of the extension footing.

The template method should not be used on very long spans or where there is a very large difference in the elevations of supports, or if used in such cases should be checked by direct computation based on the span in question.

97. Miscellaneous. When long and short spans are adjacent, and a suspension insulator is used to support the conductor between the spans, low temperatures will tend to make the insulator swing toward the short span. Problems of this type can be solved by equating moments about the insulator support.

Uplift may occur in cold weather on a suspension insulator supporting a conductor at a point between two spans on a mountainside. To determine whether or not there will be uplift, the two spans can be computed as one, ignoring the middle insulator, and if the curve passes below the insulator there is no uplift; if above, there is uplift. There is danger in the latter case of the insulator either swinging around or crumpling up and allowing the conductor to become short-circuited against the tower. Such situations may be cured by changing the tension or tower location, by counterweighting or, most commonly, by using two strings of insulators instead of one, giving dead-end support. Increasing the height of the tower a few feet may cure the trouble.

The wind pressure on cylindrical wires is usually determined by the empirical formula,

$$\text{Pressure} = 0.0025\, V^2 \text{ lb per sq ft projected area,}$$

in which V is the wind velocity in miles per hour. The presence of ice, by increasing the projected area, materially increases the wind pressure on a line. Heavy storm loading is defined as that caused by a half-inch radial thickness of ice combined with a wind pressure of 8 pounds per square foot of projected area. This pressure corresponds to a wind velocity of about 57 miles per hour. Greater wind velocities than this could probably be depended on to free the conductors of ice. In determining the worst expected conditions of ice, wind and low temperature, it is probably safe to use 0 degrees F even though the actual temperature may go lower, because the ice would probably be shaken off in a heavy wind in the time required for the temperature to drop to zero. The ice formation occurs only at and near 32 F.

In general the heavy loading must be provided in the United States as far south as Tennessee and Oklahoma in the eastern half of the country. All possible data pertaining to sleet and wind storms in the section to be traversed should be collected to assist in planning. The Bell System Sleet Map (see reference at end of chapter) and United States Weather Bureau reports are valuable sources of information for lines in the United States.

Spacing of conductors should be such as to provide safety against flashover when the wires are swinging in the wind, and when the ice coating of one wire falls off before it does on others, causing differences in sag. The proper spacing is a function of span length, voltage and weather conditions. The use of horizontal spacing eliminates the danger caused by unequal ice loading. Small wires or wires of light material are subject to being whipped about by the wind more than heavy conductors, and, other things being equal, they should be given greater spacings. A good idea of the spacing which has been used on lines may be obtained from a perusal of the details of transmission lines in the United States (see reference at end of chapter).

Ground wires, which are firmly attached to towers, aid in increasing the strength and rigidity of the transmission structure through their action as guy wires. Corner towers are usually guyed so as to resist better the unbalanced pull of the conductors.

Conductor vibration. One type of conductor vibration already mentioned is that caused by the formation of eddies in the air on the lee side of the wires. These vibrations are of relatively high frequency (of the order of 10 to 50 cps) and low amplitudes.

A completely different type of vibration, sometimes called " galloping " or " dancing," is caused by the action of a fairly strong wind on a wire covered with ice when the ice coating happens to take a form which makes a good air-foil section. Then the whole span may sail up like a kite until it reaches the limit of its slack, stops with a jerk, and falls or sails back. Such violent motions have been observed to continue for hours, with amplitudes of several feet, and frequencies of the order of one cycle per second.

Still another type of conductor vibration, associated with corona and a very light wind, has been observed, but this type is not so troublesome as the other two.

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PROBLEMS ON CHAPTER IX

Prob. 1–9. A 1000-ft span of no. 0000 hard-drawn copper wire, weighing 0.641 lb per lin ft and having an allowable tension of 4150 lb due to dead load, is to cross a certain river. How high must the points of support be above high-water level, if the clearance is to be 50 ft at the time of maximum stress?

Prob. 2–9. Work problem 1–9 if there is a $\frac{3}{4}$-in. coating of ice on the wires, making the total dead load weight 1.770 lb per lin ft.

Prob. 3–9. A transmission line over level country is to be designed with a known horizontal tension, and it is desired to find the most economical length of span. The minimum clearance is to be c ft, the horizontal tension H lb, and the weight per unit length is w lb. Find the expression for the most economical spacing on the assumption that the cost of towers, complete, varies directly as the square of the height of the point of support, being k times the square of the height in feet.

Prob. 4–9. A 1000-ft span of copper wire has a sag of 50 ft. What is the actual length measured along the catenary? What is the unstressed length?

Prob. 5–9. Owing to an increase in temperature, the sag of the span of problem 4–9 increases to 52 ft. What is the new length of arc?

Prob. 6–9. What is the horizontal stress in the span of problem 4–9?

Prob. 7–9. What is the horizontal stress in the span of problem 5–9?

Prob. 8–9. The modulus of elasticity of the copper wire of problem 5–9 is 16,000,000 lb per sq in. (*a*) What was the decrease in length due to the decreased tension? (*b*) What was the net increase in length? (*c*) What was the increase attributable to the change in temperature directly? (*d*) What was the amount of change in temperature?

Prob. 9–9. Work the example of article 92 for a span of 2500 ft.

Prob. 10–9. Work the example of article 92 if one point of support is 100 ft higher than the other. Span is 2000 ft.

Prob. 11–9. What is the maximum possible span for a uniform wire with strength S and weight w per unit length? This is to be worked with no limitation on the permissible sag. What is the sag for maximum span? At what angle does the wire leave the supports? Supports are at equal elevations.

Prob. 12–9. Compute the curves of figure 118 for a 1200-ft span.

Prob. 13–9. Find the sag in the Mississippi river span described in the example of article 93, when there is maximum ice loading and no wind, at 32 F.

CHAPTER X

CORONA AND INSULATORS

98. Appearance of Corona. When an alternating voltage is impressed between two or more parallel wires whose spacings are large compared with their diameters, there is no apparent change in the condition of the air surrounding the wires if the voltage is low. If the voltage is gradually increased, however, eventually a point is reached at which the air close to the conductors becomes luminous, giving out a faint violet light. At the same time a hissing noise can be heard, and the characteristic

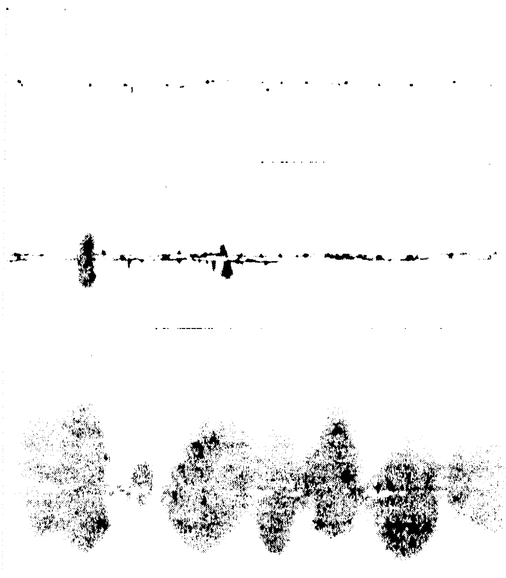

FIG. 120. Corona on one of two 61-strand 600,000-cir-mil copper cables at 12.5-ft spacing. $f = 60$ cps.
Top figure, 423,000 volts. Middle figure, 511,000 volts. Bottom figure, 734,000 volts.
General Electric Company.

odor of ozone can be detected. If the voltage is raised still further, these phenomena become more pronounced, the luminous region increasing in both size and brightness. If the conductors are rough or dirty, usually the brightest parts of the luminous envelope are near the rough or dirty spots.

Photographs of corona formation are shown in figure 120.

The term **corona** is used to designate in particular this visible luminous envelope, and in general all the phenomena accompanying its formation.

Corona is accompanied by a liberation of heat, and a wattmeter connected in a circuit having corona will indicate that power is being supplied to the corona.

If the voltage impressed on the wires is direct instead of alternating, there is a difference in the appearance of the positive and negative wires. The positive wire has a uniform glow about it, while the glow about the negative wire is more spotty, and even sometimes in the form of short streamers from any rough or sharp spots on the wire. These same phenomena can be observed with alternating voltage if the wires are viewed through a stroboscope or synchronous shutter, so that the wires are seen only when they are at a positive potential, or only when at a negative potential.

If the difference of potential between wires is increased sufficiently, the corona envelopes will grow larger and larger until finally there is a spark or arc between them. If the spacing is only a few times the diameter of the wires, the spark or arc may pass before any corona is formed.

99. Explanation of Corona in Terms of the Electron Theory. At all times the air is ionized to a slight extent; that is, in addition to the normal uncharged molecules of the different gases, there are present other molecules of these gases which have lost one or more electrons from their normal quota, leaving them with net positive charges. When an electric gradient is set up in air, all ions and free electrons which are present in the field have forces acting upon them proportional to the product of the electric field intensity or gradient multiplied by the net charge on the ion or electron. This force accelerates the ions or electrons, and their acceleration would be continuous were it not for the numerous collisions of these charged particles with one another and especially with the uncharged molecules, which may be in comparison almost motionless. The velocity which the charged particles may reach depends to a large extent, then, on how far they go without being stopped or at least deflected by collisions with other molecules. The distance which any one charged particle will go without experiencing a collision depends on chance, but where there are millions and billions of charged particles all obeying the same laws and moving under the same average conditions, it is possible to calculate the *average* distance one of them will move before striking another. This average distance of motion is called the **mean free path,**[1] and its value depends upon the size of the molecules and upon the temperature and pressure, in so far as they affect the number of molecules per unit volume.

The velocity which the average charged particle has at the time of a collision depends on its charge, its mass, its mean free path and the potential gradient which was acting upon it during its motion. If these values are such as to give it a sufficiently high velocity, the charged particle will, upon striking another molecule, dislodge from this molecule one or more electrons. This produces another ion and one or more free electrons, which in turn are accelerated until they collide with other molecules, thus producing other ions. The process of ionization may thus be cumulative in its action, and, if the potential gradient is great enough to cause the cumulative effect, either corona or a spark is the result. The ionization of the air causes a redistribution of the potential gradient, and if the redistribution is such that the gradient over the remaining portion of air between the conductors of different polarity is increased above the breakdown gradient of air, a spark or flashover results. If the redistribution lowers the maximum potential gradient below the breakdown gradient except in the immediate neighborhood of the conductors, only the air around them is broken down, and corona envelopes are formed. The remainder of the air at a distance from the conductors retains its original insulating properties.

The uniform potential gradient which is necessary to cause cumulative ionization in air at a temperature of 25 C and a pressure of 760 mm of mercury (these represent standard conditions) is 30 kilovolts per centimeter. A gradient of exactly this magnitude at the surface of a wire would not, however, be sufficient to cause corona, because, first, the gradient diminishes as the distance out increases, and so the average gradient moved through up to the time of the first collision is less than the critical gradient. The mean free path is so short in comparison with the radius of any transmission line conductor, however, that this effect is negligible. A second and much more important effect is that the ions do not have a chance to accumulate in sufficient quantities to produce a noticeable effect unless the critical gradient is exceeded for a considerable distance, so that the ions existing very near the wire may multiply, by repeated collisions, many times over.

[1] The **mean free path** would have a very definite meaning if molecules and ions could be considered as elastic spheres which sustain actual collisions with one another. When it is considered that most of the deflections are probably caused by the mere repulsion between like charges in close proximity, rather than by collisions, it becomes evident that there can be no very definite natural dividing line between these quasi-collisions and free motion. Nevertheless, the idea of mean free paths is a very useful one, especially in an elementary treatment.

100. Empirical Laws of Corona, Applicable to Two-Wire Single-Phase and Three-Wire Equilateral Three-Phase Lines. The laws governing corona formation are given by the following empirical formulas, first determined by F. W. Peek, Jr.:

$$p = \frac{390}{\delta}(f + 25)\sqrt{\frac{r}{D}}(e - e_d)^2\, 10^{-5} \text{ kw loss per mile of conductor} \qquad (426)$$

in which δ is the relative density of the air, equal to

$$\delta = \frac{3.92\, b}{(273 + T)} = 1 \text{ under standard conditions} \qquad (427)$$

$$(b = 76.0,\ T = 25 \text{ C}).$$

b = barometric pressure in centimeters of mercury.
T = temperature in degrees C.
f = frequency in cycles per second.
r = wire radius in inches. It is the radius of the circumscribing circle in a stranded cable.
D = interaxial spacing in inches for single-phase or equilateral three-phase lines.
e = *effective* voltage to neutral, in kilovolts. It is the line voltage divided by 2 for single phase; by $\sqrt{3}$ for balanced three phase.
e_d = disruptive critical voltage, effective value, given by

$$e_d = 123\, m_0 r \delta \log_{10} \frac{D}{r} \text{ kv to neutral} \qquad (428)$$

in which r is again in inches, and m_0 is an irregularity factor equal to

m_0 = 1.00 for polished wires
0.98 to 0.93 for roughened or weathered wires
0.87 to 0.83 for seven-strand cables, concentric lay
0.85 to 0.80 for 19-, 37- and 61-strand cables, concentric lay.

When cables greater than $\frac{1}{2}$ inch in diameter are used the seven-strand cable becomes impracticable. The larger strands become mutilated in manufacture, and the irregularity factor is lower than on cables of a greater number of strands. For *new* concentric-lay cable 1 inch in diameter (approximately the size used on 220 kv) m_0 is about the same for 19-, 37- and 61-strand cables and varies from 0.80 to 0.85. This factor probably improves as the conductor becomes weathered and the roughness and points are oxidized away. The corona brushes tend to do this. A concentric-lay aluminum cable of the same size was found to have an m_0 factor somewhat higher than 0.85 after operating for several years. This seems a safe value to use for large concentric-lay cables.

In making special cables to obtain an increased diameter with a given amount of material, care must be taken not to make a design of irregular outline or an outline with projections; otherwise there may be a loss instead of a gain.

For approximating storm losses, e_d should be taken as 0.8 of its value as given by formula (428) for fair weather.

The value of δ, given by equation (427), is unity under standard conditions of 76.0 cm of mercury pressure, and at 25 C temperature.

Formula (426) for power loss holds very accurately above a certain critical voltage e_v, called the visual critical voltage, at which corona appears all along the line. e_v is higher than e_d, and its value is given by

$$e_v = 123\ m_v\ \delta r\left(1 + \frac{0.189}{\sqrt{\delta r}}\right) \log_{10} \frac{D}{r} \text{ rms kv to neutral} \qquad (429)$$

in which m_v is another irregularity factor, equal to

m_v = 1.00 for polished wire
0.72 for local corona all along cable
0.82 for decided corona all along cable.

For voltages lying between e_d and e_v the corona loss is generally less than that calculated for small wires and greater than that calculated for large, new cables. For weathered cable in good condition it usually checks the calculated curve. It depends to such a great extent on the exact value of the irregularity factor that the accuracy obtainable from the formula is poor if the factor is not known precisely.

If the wires are small, the following formula is more nearly correct:

$$p = \frac{390}{\delta}(f + 25)\sqrt{\left(\frac{0.93}{D^2}\right) + \frac{(r + 0.016)}{D}}\ (e - e_d)^2\ 10^{-5} \text{ kw per mile of conductor.} \qquad (430)$$

The disruptive critical voltage e_d is equal to

$$e_d = g_d r \ln \frac{D}{r} \text{ effective kv to neutral.} \qquad (431)$$

The symbol g_d is the disruptive gradient, given by

$$g_d = g_0 m_0 \delta\left[1 + \frac{0.188}{(1 + 1485\ r^2)\sqrt{\delta r}}\right] \text{ effective kv per in.} \qquad (432)$$

g_0 = 53.6 effective kv per in. for wires of any practical line.

Formulas (430), (431) and (432) hold for conductors 0.010 inch and more in radius, and with spacings of 6 inches and more. Formulas (426) and (428) are simpler expressions which give approximately the same results for most practical sizes and spacings.

101. Corona Calculations on Unsymmetrical and Multicircuit Lines. Equation (426) may be expressed in terms of gradients, instead of voltages, as follows:

$$p = \frac{2070}{\delta}(f + 25)\sqrt{\frac{r}{D}}\left(\log_{10}\frac{D}{r}\right)^2 r^2(g - g_d)^2\, 10^{-5} \text{ kw}$$

per mile of conductor (433)

by making the substitution of $gr \ln \frac{D}{r}$ for e, the voltage to neutral, and $g_d r \ln \frac{D}{r}$ for e_d, the disruptive voltage to neutral. These substitutions are valid for the two-wire single-phase and equilateral three-phase cases, and in order to make application of the formulas to unsymmetrical and multicircuit lines it must be noticed that the formation of corona depends fundamentally upon the gradient rather than upon the actual voltage. Wherever it is practicable, the formulas are given in terms of voltage, because the amount of computation necessary is thereby reduced, but it is necessary, in making calculations for irregular and complicated arrangements of parallel wires, to calculate the gradient, and substitute it in equation (433). If the gradients are different around the different wires, a separate calculation must be made for the loss around each of the wires, and the sum taken for the total loss. No experimental data are available concerning the correct value of D to be used, and there is no theoretical basis for choosing any particular value. However, the expression for loss in (433) is only slightly affected by a considerable change in D, and the use of a mean value will probably give satisfactory results.

But few useful and dependable data are available for various unsymmetrical configurations. Many papers may be found in the technical literature giving the results of corona loss tests made on actual long lines, but a great deal of caution is necessary in making general use of these. In some cases the authors have rushed into print with new formulas and laws based upon sadly insufficient data, some of them even being based upon measurements made principally between e_d and e_v, between which values the careful work of Peek, Whitehead and others has shown that considerable irregularity is inevitable. On long lines, too, the voltage varies considerably from point to point, as do the temperature and pressure and perhaps other weather conditions. It is fairly safe to say that short experimental lines must be looked to for the establishment of dependable laws. It seems certain that the losses in the unsymmetrical cases can be calculated very closely from (433), but the results will not be as accurate as for the symmetrical spacing because the equipotential surfaces around the conductors will have different shapes, and in general the corona envelope exterior will be one of the equipotential surfaces. It is not likely that the error due to this difference is large, because in practice corona is kept within small limits, and the equipotential surfaces near the wires are nearly circular, no matter what the configuration, as long as the spacing is large compared with the radius.

102. The Calculation of Potential Gradient. The potential gradient at the surface of the conductors in ordinary overhead lines may be calculated with very little error by neglecting the gradient set up by any charges except that on the conductor at whose surface the gradient is being calculated. If this assumption be made, with the further assumption that the charge on the conductor is uniformly distributed over the surface, the calculated gradient, as shown in Chapter IV, is $\frac{18 \times 10^9 q}{r}$ volts per meter, q being the charge in farads per meter length of line, and r the radius of the wire in meters. The calculation of potential gradient is therefore almost identical with the problem of the calculation of charge, which was outlined in Chapter IV. It is not necessary, however, to do all the preliminary work of finding the charge. To illustrate a shorter and more convenient method, let it be required to compute the maximum gradients (effective values) for the line shown in figure 22, page 41, with a balanced three-phase impressed voltage of E rms kilovolts between wires and no transposition or grounding. Let g_a be the vector rms gradient in kilovolts per inch at the surface of wire a, g_b at b and g_c at c. Then the gradient due to wire a at a distance x inches out from its center is $r\frac{g_a}{x}$ kilovolts per inch, and similar expressions hold for the gradients set up by the other two wires. The voltage between any two wires is equal to the sum of the integrals of the three gradients between the wires. Also, if the sum of the three vector charges is zero and the three wires are all the same size, the sum of the three vector surface gradients is zero.

$$E_{ab} = E + j\,0 = r\left[(g_a - g_b)\ln\frac{D_{ab}}{r} + g_c \ln\frac{D_{bc}}{D_{ac}}\right] \text{ vector kv.} \qquad (434)$$

$$E_{bc} = -0.5\,E + j\,0.866\,E = r\left[(g_b - g_c)\ln\frac{D_{bc}}{r} + g_a \ln\frac{D_{ca}}{D_{ba}}\right]. \qquad (435)$$

$$g_a + g_b + g_c = 0 \text{ vector kv per in.} \qquad (436)$$

These three equations do not lead to simple expressions for the three gradients if solved in their present form, but are readily solvable when numerical values are known for the distances.

In connection with these calculations there should be noted a principle which is likely to lead to error if studied in a cursory manner. The corona at any point on a transmission line depends on the gradient (and hence the charge per unit length) of the line *at that point*, and if the spacing is irregular, no amount of transposition can equalize the charges of the three wires at any given point. Therefore, in the calculation of gradient, the procedure is the same whether the line is transposed or not. Transposition has the effect of equalizing the amount of corona loss per phase, on the average, along the whole length of the line.

103. Sample Calculation of Corona Loss.

Line — three no. 0000 solid wires equilaterally spaced 10 ft.
Frequency — 60 cps.
Voltage — 139,000 between lines, balanced three phase.
Condition of wire surface — smoothly weathered.
Weather — fair, with temperature of 20 C and barometric pressure 72.2 cm.

Solution for loss per mile.

$$\delta = \frac{3.92 \times 72.2}{293} \quad 0.966.$$

$$e_d = 123\, m_0 \delta r \log_{10} \frac{D}{r} = 123 \times 0.96 \times 0.966 \times 0.23 \times \log_{10} \frac{120}{0.23}$$

$$= 71.1 \text{ effective kv to neutral.}$$

$$e = \frac{139}{\sqrt{3}} = 80.0 \text{ effective kv to neutral.}$$

For a mile of the whole line, all three wires, the loss is

$$3\,p = 3 \times \frac{390}{\delta} (f + 25) \sqrt{\frac{r}{D}} (e - e_d)^2\, 10^{-5} \text{ kw}$$

$$= 3 \times \frac{390}{0.966} (85) \sqrt{\frac{0.23}{120}} (80 - 71.1)^2\, 10^{-5}$$

$$= 3.6 \text{ kw per mile.}$$

104. Effect of Atmospheric Conditions on Corona. The effect of variations in temperature and barometric pressure is taken into account in the formulas by the density factor δ. There is no appreciable effect due to high winds or to humidity. Weather conditions which cause deposits of moisture or ice on the wires, however, considerably increase the loss due to corona over the fair-weather value of loss for the same temperature and barometric pressure. Moisture (dew), frost, fog, sleet, rain and snow all have large effects, both in increasing the loss and in reducing the critical voltage. Smoke also increases the loss and lowers the critical voltage. Sleet formation is not prevented by high voltage.

Peek states that the storm loss may be approximated by using for e_d eight tenths of its normal value.

105. Effect of Corona on Line Design. Corona has many disadvantages, and though it may be questioned whether or not any positive advantages are produced by its actual formation in normal operation, there undoubtedly are certain advantages attendant upon line designs which result in corona formation, as compared with designs which would avoid it. In the correct design of a high-voltage line, therefore, a balance should be struck between the advantages and the disadvantages.

It is generally desirable to avoid corona loss for all fair-weather conditions. This can ordinarily be done without great expense because the power transmitted usually requires a conductor large enough to meet this condition. If the condition is not met in this way an aluminum conductor with a steel core or a hollow copper conductor can be used.

In addition to the disadvantage of the power loss caused by corona, there is another disadvantage. The leakage current accompanying corona flows only during the peak periods of the voltage waves, and so this current contains harmonics in exceptionally large proportion. Harmonics are particularly troublesome in causing inductive interference with parallel communication circuits.

When very high-voltage traveling waves are set up by lightning, corona is effective in producing more rapid attenuation.

Ozone is produced by corona, and may cause deterioration in any organic material near by.

No categorical design rules can be given regarding corona, except the general principle of design for minimum total annual cost, interpreted in a broad way. Each line presents a problem in itself different from all others, and two apparently identical transmission projects in different parts of the country might require different treatments of the corona problem, on account of different climatic conditions. In a region where there are only a few storms each year on the average, it might be wise to allow considerable corona loss on these rare occasions, so as to enjoy the continuous benefit of the higher voltage without increasing too much the cost of line construction.

106. Distribution of Voltage Stress on Suspension Insulators. High-tension insulators are usually built up of several porcelain disks in series, the disks being connected by steel hardware. A sketch of a typical string insulator is shown in figure 121. If all the units or disks were exactly alike, such an insulator string would have across each unit the same value of voltage, *if the voltage impressed were direct.* A certain leakage current flows through the insulator, and the direct voltage across each unit would be the leakage current multiplied by the resistance of each unit.

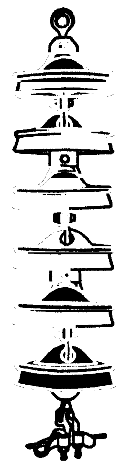

FIG. 121. Construction of suspension insulator.

When the impressed voltage is alternating, the voltages across the different units do not have the same values, because the charging current is not the same all through the string. In addition to the capacitance from each connector to the next, there is a capacitance from each connector to ground, and to the line wire. In addition to these, there are capacitances from each connector to the others which are not adjacent, but separated by two or more units.

In order to develop an insight into the problem of potential stresses and means for adjusting them, let us consider the effects, one by one, of the various capacitances. The leakage is usually small enough so that at 60 cps it has no appreciable effect on the potential distribution.

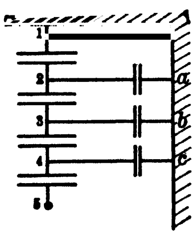

FIG. 123. Capacitances between adjacent connectors, and to tower.

If we consider only the capacitance of each unit, from one connector to those adjacent on each side, it is evident that the calculated voltage stress would be the same on each unit. See figure 122.

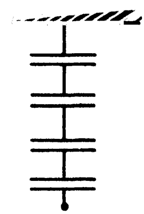

FIG. 122. Capacitances between adjacent connectors of insulator string.

Suppose we now bring our analysis nearer to the actual condition of affairs by taking into account the effect of the various capacitances to ground. In figure 123, let the capacitance of each unit be C farads, and from each connector to tower and ground $C/4$ farads. We wish to find the percentage of total voltage across each unit.

Solution. Let the voltage across the upper unit 1–2 be designated by v. This is also the voltage across condenser 2–a.

Current of 1–2	$j\omega C \times v$	
Current of a–2	$j\omega C \times \frac{1}{4}v$	
Current of 2–3	$j\omega C \times \frac{5}{4}v$	
Voltage across 2–3		$\frac{5}{4}v$
Voltage across b–3		$\frac{9}{4}v$
Current of b–3	$j\omega C \times \frac{9}{16}v$	
Current of 3–4	$j\omega C \times \frac{29}{16}v$	
Voltage across 3–4		$\frac{29}{16}v$
Voltage across c–4		$\frac{65}{16}v$
Current of c–4	$j\omega C \times \frac{65}{64}v$	
Current of 4–5	$j\omega C \times \frac{181}{64}v$	
Voltage across 4–5		$\frac{181}{64}v$

$$\text{Total voltage wire to ground} = \left(1 + \frac{5}{4} + \frac{29}{16} + \frac{181}{64}\right)v = \frac{441}{64}v.$$

There is, according to this calculation,

$\frac{64}{441} =$ 14.5% across the top unit

$\frac{80}{441} =$ 18.1% across the second unit

$\frac{116}{441} =$ 26.3% across the third unit

$\frac{181}{441} =$ 41.1% across the bottom unit

$\frac{441}{441} =$ 100.0% total.

The top unit takes about one-seventh of the total voltage, and the bottom one more than two-fifths. If the string is longer, the inequality of voltage becomes even more marked. For instance, in a 6-unit string with capacitances as just given for each unit, the calculated distribution is:

5.3% across top unit
6.6% across second unit
9.6% across third unit
15.0% across fourth unit
24.2% across fifth unit
39.3% across bottom unit

100.0%

The addition of two more units apparently decreases the stress across the bottom unit but very little.

Now consider the effect of the capacitances between the various connectors and the line conductors. We will take the capacitances of the units and from connectors to ground the same as before, but in addition let the capacitance from each connector to the line wire be $C/8$ farads. The connection is shown in figure 124. We wish to find again the percentage of total line voltage across each unit.

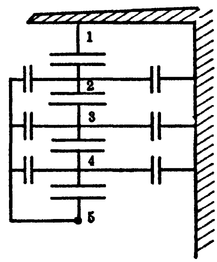

FIG. 124. Capacitances between adjacent connectors, and from connectors to tower and to conductor.

Solution. Let the voltage across the top unit be designated by v, and that across the whole string by V.

$$V_{12} = v.$$

$$I_{12} = j\omega C v.$$

$$I_{23} = j\omega C v + j\omega v \frac{C}{4} - j\omega(V - v)\frac{C}{8}$$

$$= j\omega C\left(\frac{11}{8}v - \frac{1}{8}V\right).$$

$$V_{23} = \frac{11}{8}v - \frac{1}{8}V.$$

$$I_{34} = j\omega C\left(\frac{11}{8}v - \frac{1}{8}V\right) + j\omega\left(\frac{19}{8}v - \frac{1}{8}V\right)\frac{C}{4} - j\omega\left(V - \frac{19}{8}v + \frac{1}{8}V\right)\frac{C}{8}$$

$$= j\omega C\left(\frac{145}{64}v - \frac{19}{64}V\right).$$

$$V_{34} = \frac{145}{64}v - \frac{19}{64}V.$$

$$I_{45} = j\omega C\left(\frac{145}{64}v - \frac{19}{64}V\right) + j\omega\left(\frac{297}{64}v - \frac{27}{64}V\right)\frac{C}{4} - j\omega\left(V - \frac{297}{64}v + \frac{27}{64}V\right)\frac{C}{8}$$

$$= j\omega C\left(\frac{2051}{512}v - \frac{297}{512}V\right).$$

$$V_{45} = \frac{2051}{512}v - \frac{297}{512}V.$$

$$V_{15} = V_{12} + V_{23} + V_{34} + V_{45}$$

$$= \frac{4427}{512}v - \frac{513}{512}V = V.$$

$$v = V_{12} = \frac{1025}{4427}V = 0.232\ V.$$

V_{23} =	0.193 V
V_{34} =	0.228 V
V_{45} =	0.347 V
Total	1.000 V

It is seen that the minimum voltage does not come on the top unit, but on one lower down. The maximum voltage is on the lowest unit, as before.

In figure 125 are shown experimental curves of distribution on strings of lengths ranging from 4 to 16 units, without shielding.

Figure 126 shows a comparison of the voltage distributions with and without shielding, on a 16-unit string.

Figure 127 shows the heavy rings and horns used for shielding on the 220,000-volt Big Creek line of the Southern California Edison Company. They are also of value in that when an arc occurs it will usually jump between rings or horns, and stay clear of the insulator string. Also, an arc between the conductor and the lower structure might result in the conductor being melted and dropping to the ground. Typical flashovers on insulators with and without shields are shown in figures 128 and 129.

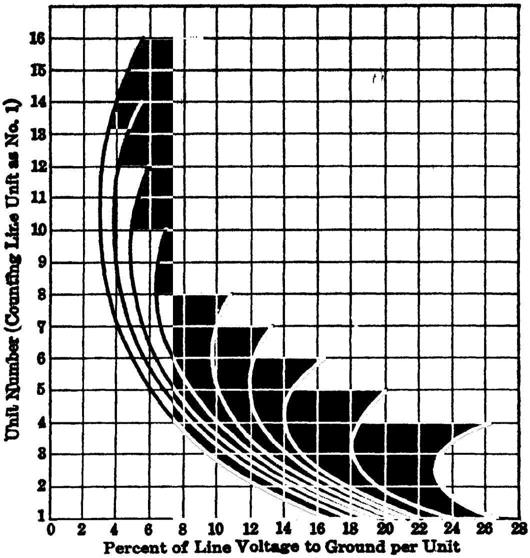

FIG. 125. Voltage distributions on unshielded string insulators having from 4 to 16 units. — *Locke Insulator Corporation.*

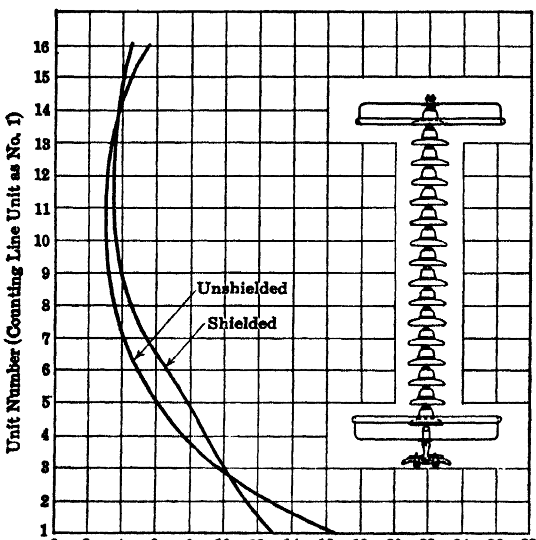

FIG. 126. Voltage distributions on a 16-unit insulator string with and without shielding. — *Locke Insulator Corporation.*

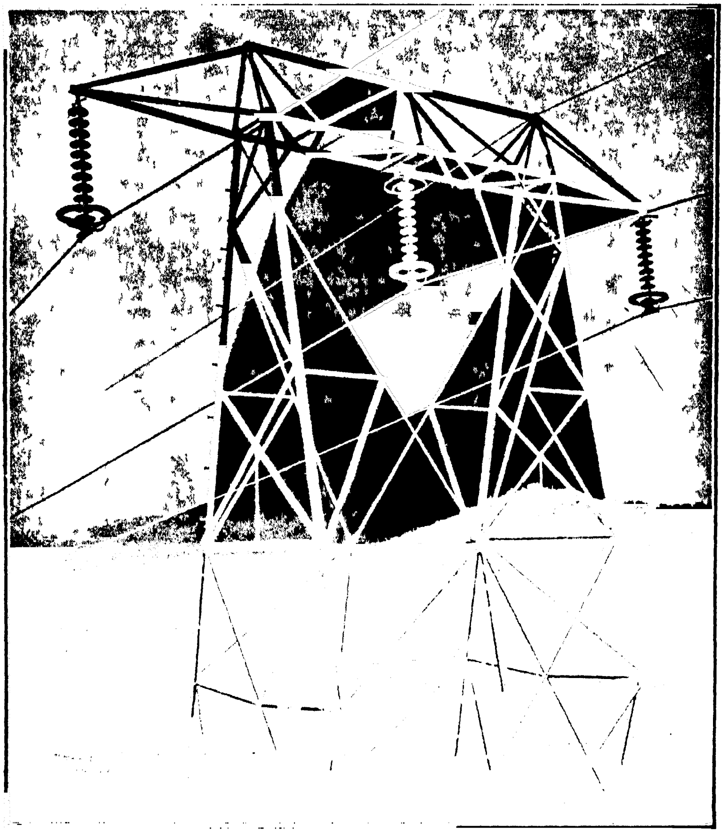

FIG 127 Standard suspension tower on the 220,000-volt Big Creek line of the Southern California Edison Co, showing guard rings and arcing horns —*Aluminum Company of America.*

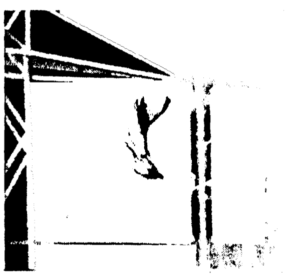

FIG. 128. Dry 60-cps flashover on a suspension insulator shielded with a ring shield. Arc clears string —*General Electric Company.*

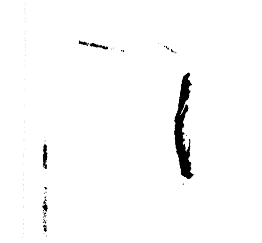

FIG. 129. Dry 60-cps flashover on an unshielded suspension insulator. Arc cascades badly. —*General Electric Company.*

107. Live-Line Testing of Insulators. A very convenient and quick method of testing for defective units in string insulators has been developed by T. F. Johnson (see reference at end of chapter), which depends on the stress distribution as worked out for a normal string. The process involves the use of a tool called a buzz stick, and two distinct testing procedures, illustrated in figures 130 and 131. The buzz stick has a two-pronged metal fork, with a ball at the end of one prong. The fork is mounted on a long wooden handle, which should be tested frequently for its insulating properties.

In the first test, figure 130, the ball only is touched to the line conductor, and drawn slowly away. This is repeated at the bottom insulator cap, the second cap, and so on to the top. Corona will form, and produce a buzzing sound, the intensity being approximately pro-

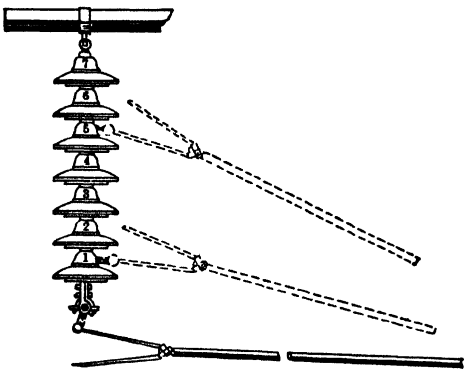

FIG. 130. Method of making preliminary test with buzz stick. — *T. F. Johnson.*

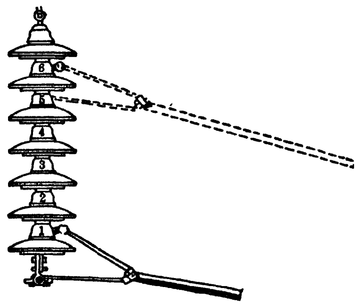

FIG. 131. Method of making final test with buzz stick. — *T. F. Johnson.*

portional to the difference in the currents through the two adjacent units. The buzz will be loudest at the bottom of the string, and will fall to zero about two-thirds of the way up. There will be noise again at the top units, above the silent position. In case one or more units are defective and are not taking their share of the total voltage, there will be a distortion of the normal pattern of noise.

If the first test indicates that the string seems to be in good condition, the second test is made. This is shown in figure 131. The sharper prong is touched to the conductor and the ball then brought into and out of contact with the cap above the bottom unit, short-circuiting that unit. A spark is produced, and the intensity of its sound is used as a measure of the voltage across the unit. The second unit is tested in the same way, and so on up the string. A normal string should yield sparks whose sounds are greatest for the bottom unit, decrease to a minimum but not to zero about two-thirds of the way up, and then increase to the top. A defective unit will show diminished noise or none at all.

It would be dangerous to make this short-circuiting test without first making the preliminary test, because a flashover might result if there were already two or three defective units in the string and then a good unit was short-circuited in the course of the test.

Some power companies replace the defective units while the line is live; others do live-line testing but remove the voltage from the line for repairs and replacements.

REFERENCES

F. W. Peek, Jr., *Dielectric Phenomena in High Voltage Engineering.*

L. V. Bewley, *Traveling Waves on Transmission Systems.*

T. F. Johnson, Jr., *Buzz Stick Method for Locating Defective Insulators*, U. S. Patent 1,366,078.

PROBLEMS ON CHAPTER X

Prob. 1–10. At what voltage between lines will corona start on a three-phase line with equilateral spacing, the three conductors being no. 0000 solid copper spaced 8 ft between centers? Atmospheric conditions are standard ($b = 76$ cm of mercury, $T = 25$ C), and the irregularity factor m_0 is 0.95.

Prob. 2–10. Work problem 1–10 for 1,000,000-cir-mil stranded copper (61 strands, concentric lay) spaced 15 ft between centers in an equilateral triangle. $m_0 = 0.83$.

Prob. 3–10. What is the line voltage required to cause corona on a three-phase line of three no. 00 solid conductors with flat spacing, 6 ft between centers of adjacent wires? $m_0 = 0.95$, $\delta = 1$.

Prob. 4–10. A line of three no. 000 solid wires has an equilateral spacing of 8 ft. What is the corona loss per mile at 140 kv between lines? At what line voltage will the loss be 5.0 kw per mile?

The temperature is 30 C and the barometric pressure 71.0 cm of mercury. $m_0 = 0.95$. $f = 60$ cps.

Prob. 5–10. A certain three-phase equilaterally spaced transmission line has a total corona loss of 53 kw at 106,000 volts and a loss of 98 kw at 110,900 volts. What is the disruptive critical voltage between lines? What is the corona loss at 113,000 volts?

Prob. 6–10. A double-circuit three-phase line has its six equal conductors arranged at the corners of a regular hexagon, 8 ft on a side, the two wires of each phase being diametrically opposite each other. The wires are 37-strand concentric lay copper of 250,000 cir mils, the irregularity factory m_0 being equal to 0.85. The density factor δ is unity. Find the critical corona-forming voltage.

Prob. 7–10. The line of problem 5–10 has to deliver a power of 10,000 kw at unity power factor. The total resistance of each phase of the line is 9.8 ohms. Calculate the load voltage which would result in the minimum line loss, neglecting the effect on corona of any change in voltage along the line.

Prob. 8–10. Work problem 7–10 for a load of 10,000 kw at 0.7 power factor lag.

Prob. 9–10. How far apart must cables 1 in. in diameter ($m_0 = 0.83$) be spaced equilaterally so as just to be at the disruptive critical voltage when the line-to-line voltage is 220 kv? $\delta = 1$.

Prob. 10–10. Work problem 9–10 for no. 0000 solid wire, 110,000 volts. $m_0 = 0.95$, $\delta = 1$.

Prob. 11–10. A single-circuit three-phase line has its three 1.036-in.-diameter conductors arranged in a horizontal plane 25 ft on centers and 40 ft average above ground. Find the critical corona-forming voltage, assuming $m_0 = 0.85$ and $\delta = 1.00$. The neutral is at ground potential.

Prob. 12–10. The line of problem 12–10 operates with neutrals of transformers not grounded. Through accident one of the outside line conductors becomes dead grounded. The three line voltages remain in approximate balance. What magnitude of line voltage would just cause corona on the middle wire? On the ungrounded outside wire?

Prob. 13–10. If the line voltage is 220,000 on the grounded line of problem 12–10, what will be the corona loss per mile on each of the wires?

Prob. 14–10. In figure 123, the capacitance across each unit is C farads, and from each connector to ground $C/3$ farads. Calculate the voltage across each unit expressed as a percentage of the total voltage.

Prob. 15–10 Work problem 14–10 for a 6-unit string.

Prob. 16–10. In figure 124, the capacitance across each unit is C; from each connector to ground $C/3$, and from each connector to the wire $C/5$ farads. Calculate the voltage across each unit expressed as a percentage of the total voltage.

Prob. 17–10. Work problem 16–10 for a 6-unit string.

Prob. 18–10. In problem 16–10, to what value would the capacitance across the bottom unit have to be increased by a guard-ring to make the voltage across it equal to that across the next higher unit? Express the voltage of each unit as a percentage of the total voltage of the string.

Prob. 19–10. Work problem 18–10 for a 6-unit string.

Prob. 20–10. In the insulator string of problem 17–10, a guard-ring brings up the value of capacitances between connectors and line conductor to $C/3$. Calculate the distribution of voltage on the string, expressed as a percentage of the total voltage.

Prob. 21–10. Plot the relative noises you would expect from both preliminary and final buzz-stick tests of the insulator string of problem 17–10 for the condition of (*a*) all units good; and (*b*) the next to the bottom unit short-circuited; (*c*) the third unit short-circuited; (*d*) the fourth; (*e*) the fifth; (*f*) the top unit.

CHAPTER XI

CALCULATION OF FAULT CURRENTS

The calculation of short-circuit currents in power systems is important for many reasons. The sizes of these currents determine the necessary ratings of circuit breakers which must interrupt them. The sizes and phase relations determine the settings and sometimes the types and locations of the protective relays which must in effect recognize and classify the different types of faults and clear them from the system. System stability is affected by short-circuit currents. There may be thermal and mechanical damage to apparatus from currents of very large magnitude. Excessive inductive interference with parallel communication circuits may be caused.

108. Symmetrical Components. The great majority of faults on power systems are of an unsymmetrical nature, the most common type being a short circuit from one line to ground. The system impedances are usually symmetrical throughout its main elements — generators, transformers, transmission lines, synchronous reactors, etc. — and the load also is usually nearly symmetrical when taken as a whole. The fault then usually imposes the only important element of dissymmetry.

The fault currents, usually unsymmetrical in the three phases, may be resolved into positive-, negative- and zero-phase-sequence currents. The magnitudes of these components are dependent on the impedances in the system, and their relative magnitudes are determined by the type of fault which occurs. It is therefore necessary, in computing fault currents, to know the impedances to each of the three symmetrical components of current offered by the different important elements in the system.

109. Transmission Line Impedances. In figure 132 are shown the positive-, negative- and zero-phase-sequence components of current in a three-phase circuit. If this circuit is a transmission line with balanced phase impedances, it makes no difference in its characteristics to reverse the phase rotation of the current. This is in fact true of any set of balanced static impedances. The negative-sequence and the positive-sequence impedances are equal, and are the same as those used in the analysis of balanced conditions.

The zero-sequence currents are in phase and equal in the three lines, and hence must have a neutral return path, such as earth or overhead ground wires. The zero-sequence resistance is equal to that of one phase plus three times the resistance of the neutral return circuit. The zero-sequence reactance is three times that of a circuit of three wires (six in a double-circuit line) in parallel as one conductor and the neutral path as the return.

It is apparent that no precise calculated value of ground return impedance is to be expected, because of varying ground conductivity.

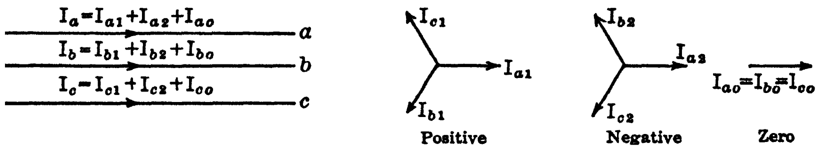

FIG. 132. Symmetrical components of current in a three-phase circuit.

However, a satisfactory determination can be made by a combination of test procedure and theory. The solution for alternating-current distribution in an assumed flat homogeneous earth acting as the return path for overhead conductors has been worked out by J. R. Carson, who has given formulas for the equivalent resistance and self and mutual reactances. J. E. Clem has published tables and charts, based on Carson's solution, in form very convenient for dealing with power transmission line problems. (See references at end of chapter.)

The impedance of a group of overhead wires whose group self geometric mean distance is D_s and whose mean height above ground is h, may be written in the following form:

$$
\begin{aligned}
z_0 &= 3\,(r_c + jx_c + z_g) \\
&= 3\left(r_c + j\,0.0007411\,\omega\log_{10}\frac{2\,h}{D_s} + z_g\right)\text{ ohms per mile,} \qquad (437)
\end{aligned}
$$

in which r_c is the resistance per mile of the line conductors, x_c is their reactance with return at the location of their images in the earth, and z_g is the term which makes correction for finite earth conductivity. At 60 cps, (437) becomes:

$$
z_0 = 3\left(r_c + j\,0.2794\log_{10}\frac{2\,h}{D_s} + z_g\right)\text{ ohms per mile.} \qquad (438)
$$

Let $z_g = r_g + jx_g$. Values of r_g and x_g for values of earth conductivity ranging from 10^{-5} mho per meter to 1.0 mho per meter are given in figures 133 and 134. Measured values of earth conductivity have ranged from 6.5×10^{-5} to 0.2 mho per meter. A fair value to be used in practice problems is 10^{-2}, but measurements should be made in solving problems dealing with an actual system. Fortunately, the ground-return impedance does not vary rapidly with variations in γ.

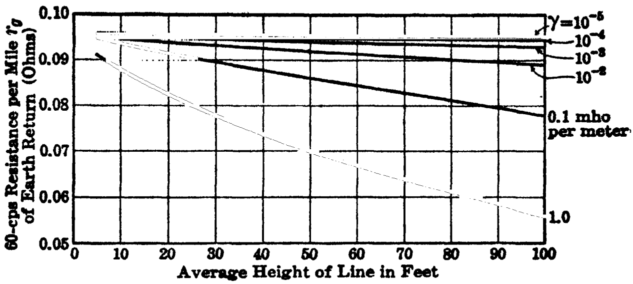

FIG. 133. Resistance per mile of earth return for various values of earth conductivity γ. — *Clem.*

Example. Find the zero-phase-sequence impedance per mile for a single-circuit transposed three-phase 60-cps line having three 300,000-cir-mil copper conductors arranged horizontally 20 ft on centers.

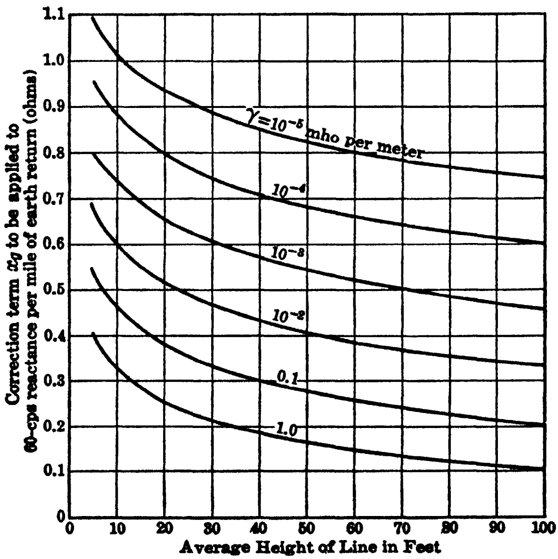

FIG. 134. Values of 60-cps reactance correction to allow for finite values of earth conductivity γ. — *Clem.*

Support points at towers are 70 ft above earth, and sag is 30 ft. Assume earth conductivity $\gamma = 10^{-2}$ mho per meter. There are no ground wires.

Solution. Average height of conductors above ground is $70 - \frac{2}{3} \times \text{sag} = 50$ ft $= 600$ in. The value of d_s for a single conductor, from table IV, is 0.242 in.

$$D_s = \sqrt[9]{0.242^3 \times 240^4 \times 480^2}$$
$$= 28.1 \text{ in.}$$

By equation (438), and using figures 133 and 134 to determine r_g and x_g, we have

$$z_0 = 3\left(\frac{0.1915}{3} + j\,0.2794 \log_{10}\frac{1200}{28.1} + 0.092 + j\,0.41\right)$$
$$= 0.468 + j\,2.60 \text{ ohm per mile per phase.}$$

For solving problems which involve coupling between lines on separate sets of towers, the original papers should be consulted.

At least about $\frac{1}{2}$ ohm of resistance occurs at each grounding connection, and this is in addition to the foregoing calculation. Three times this value will be the added zero-sequence resistance at each end.

Effect of ground wires. Ground wires have almost no effect on positive- and negative-sequence impedances, but when they are present the zero-sequence return current will divide between them and the ground path in a ratio such as to provide equal voltage drops. The circuit is analogous to the one discussed in article 27, Chapter II. The equivalent ground conductor is assumed to have the same self gmd as the parallel group of transmission conductors, and the depth of it can be determined from the reactance obtained by Carson's method. This is not an exact method, but usually gives satisfactory results.

Let r_c = resistance of power conductors in parallel, per mile.
D_{cc} = self gmd of power conductors in parallel.
r_w = resistance of ground wires in parallel, per mile.
D_{ww} = self gmd of ground wires in parallel.
D_{cw} = gmd between power conductors and ground wires.
D_{cg} = gmd between power conductors and hypothetical ground return of self gmd equal to D_{cc}.
r_g = resistance of ground path, per mile.
$I_c = 3\,I_0$ = zero sequence currents in all power conductors.
I_g = ground current.
$I_w = -I_c - I_g$ = current in all ground wires.

The drop in the ground wire at 60 cps is

$$I_w r_w + j\,0.2794\left[I_w \log_{10}\frac{1}{D_{ww}} + I_c \log_{10}\frac{1}{D_{cw}} + I_g \log_{10}\frac{1}{D_{wg}}\right], \quad (439)$$

which must equal the following expression for drop in the ground:

$$I_g r_g + j\,0.2794\left[I_g \log_{10}\frac{1}{D_{cc}} + I_c \log_{10}\frac{1}{D_{cg}} + I_w \log\frac{1}{D_{wg}}\right]. \qquad (440)$$

Solving simultaneously to find the ratio of I_w to I_c (which equals $-I_w - I_g$) we obtain

$$\frac{I_w}{I_w + I_g} = \frac{r_g + j\,0.2794 \log_{10}\dfrac{D_{wg}^2}{D_{cc}D_{cw}}}{\text{Numerator} + r_w + j\,0.2794 \log_{10}\dfrac{D_{cw}}{D_{ww}}}. \qquad (441)$$

The drop through the power conductors is

$$I_c r_c + j\,0.2794\left[I_c \log_{10}\frac{1}{D_{cc}} + I_w \log_{10}\frac{1}{D_{cw}} + I_g \log_{10}\frac{1}{D_{cg}}\right]. \qquad (442)$$

The total drop around the loop of power conductors and either ground wires or ground for return, in terms of the power conductor current I_c (which equals $3\,I_0$), is

$$I_c\left\{r_c + j\,0.2794 \log_{10}\frac{D_{cw}}{D_{cc}} + \frac{\left[r_w + j\,0.2794 \log_{10}\dfrac{D_{cw}}{D_{ww}}\right]\left[r_g + j\,0.2794 \log_{10}\dfrac{D_{wg}^2}{D_{cc}D_{cw}}\right]}{\left[r_w + j\,0.2794 \log_{10}\dfrac{D_{cw}}{D_{ww}}\right] + \left[r_g + j\,0.2794 \log_{10}\dfrac{D_{wg}^2}{D_{cc}D_{cw}}\right]}\right. \qquad (443)$$

The quantity in braces is the entire loop impedance, and three times this quantity is the zero-sequence impedance.

Even though short-circuit current calculations may be made on the basis of reactance only, neglecting resistance, nevertheless the effect of resistance r_w needs to be taken into account in (443) because it affects the value of reactance.

Example. Find the zero-sequence impedance per mile for the circuit of the previous example, to which have been added two ground wires 20 ft above the power conductors, and 20 ft on centers. Each ground wire is a "copperweld" cable of seven strands of no. 8, 30 per cent conductivity. The equivalent self gmd of each is 0.043 in., and its resistance is 1.64 ohms per mile.

Solution. From the previous example, the reactance of the circuit of three line conductors with ground return was found to be 2.60/3 or 0.867 ohm per mile. Equating this to

$$0.2794 \log_{10}\frac{D_{cg}}{28.1},$$

it is found that the equivalent distance to the hypothetical ground conductor D_{cg} is 35,500 in. or 2960 ft. Since the ground wires are 20 ft above the power conductor, D_{wg} may be taken as 2980 ft, although the difference is so small that the same figure may be used.

$$D_{ww} = \sqrt{0.043 \times 240} = 3.21 \text{ in.},$$

$$D_{cw} = 12 \sqrt[3]{(20^2 + 10^2)\sqrt{20^2 + 30^2}} = 315 \text{ in.},$$

$$D_{cc} = 28.1 \text{ in.},$$

$$r_c = \frac{0.1915}{3} = 0.0638 \text{ ohm per mile},$$

$$r_w = \frac{1.64}{2} = 0.82 \text{ ohm per mile},$$

$$r_g = 0.092 \text{ ohm per mile}.$$

Substituting into (443), the loop impedance is

$$0.0638 + j\,0.2794 \log_{10}\frac{315}{28.1}$$

$$+ \frac{\left[0.82 + j\,0.2794 \log_{10}\frac{315}{3.21}\right]\left[0.092 + j\,0.2794 \log_{10}\frac{35{,}700^2}{28.1 \times 315}\right]}{\left[0.82 + j\,0.2794 \log_{10}\frac{315}{3.21}\right] + \left[0.092 + j\,0.2794 \log_{10}\frac{35{,}700^2}{28.1 \times 315}\right]}$$

$$= 0.0638 + j\,0.294 + \frac{(0.82 + j\,0.556)\,(0.092 + j\,1.440)}{0.912 + j\,1.996}$$

$$= 0.437 + j\,0.829 \text{ ohm per mile}.$$

The zero-sequence impedance is three times this quantity.

$$z_0 = 1.31 + j\,2.49 \text{ ohms per mile per phase}.$$

Without ground wires we found, for the same line,

$$z_0 = 0.468 + j\,2.60.$$

The ground wires increased the effective zero-sequence resistance threefold, and slightly decreased the reactance. High-resistance ground wires of steel cable usually increase both resistance and reactance.

By using the expression developed by Carson for mutual impedances, an exact solution (i.e., premised on a known and uniform earth conductivity) may be obtained.

110. Transformer Impedances. Balanced three-phase transformer banks have the same impedances with reversed phase rotation, and so their positive-phase-sequence and negative-phase-sequence impedances are equal. Zero-sequence current cannot flow into the bank unless there is a neutral connection to carry away the current. There can be no zero-sequence current in a primary winding unless there is a corresponding amount in the secondary (in a two-winding transformer) so it is necessary for the other winding also to have a grounded neutral if it is Y-connected. It may be Δ-connected, as the zero-sequence current can circulate in the closed Δ. The zero-sequence impedance is the same as the impedance to positive-sequence currents, if the connection is such as to allow zero-sequence currents to flow at all. Any impedance connected in the neutral conductor has the effect of adding three times its actual value to the zero-sequence impedance, because the three equal zero-sequence currents of the three phases all flow in the neutral.

In three-winding transformers, if all three windings can carry zero-sequence current the bank may be represented by the usual equivalent

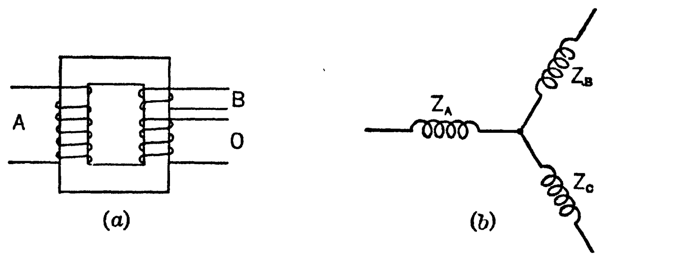

FIG. 135. Three-winding transformer and its equivalent circuit, no-load currents neglected.

circuit, magnetizing current being neglected. Referring to figure 135*a*, we can determine by test (or by calculation) the impedance Z_{AB} between windings A and B as a two-winding transformer, and similarly Z_{BC} and Z_{AC}. Then the impedances in the three arms of the equivalent circuit in figure 135*b* are given by the following expressions:

$$Z_A = \tfrac{1}{2}\,[Z_{AB} + Z_{AC} - Z_{BC}], \tag{444}$$

$$Z_B = \tfrac{1}{2}\,[Z_{AB} + Z_{BC} - Z_{AC}], \tag{445}$$

$$Z_C = \tfrac{1}{2}\,[Z_{AC} + Z_{BC} - Z_{AB}]. \tag{446}$$

These relations are easily proved by taking totals in pairs. For example, $Z_A + Z_B = Z_{AB}$.

Transformer impedances are usually in the range from 0.08 to 0.12 per unit, for large sizes.

111. Impedances of Rotating Machines. There are several reactances in synchronous machines which are important in short-circuit calculations. The initial rush of alternating current is limited by the *subtransient reactance*, but part of this current dies away very rapidly, so that it has practically disappeared in one to three cycles, before any circuit breakers have had time to open.

The *transient reactance*, which is larger than the subtransient, is the one most used in short-circuit studies. The difference between the two is due to the coupling of short-circuited paths in the rotor with the armature circuit. The time constant of the main transient component of armature current is of the order of a few seconds.

The *synchronous reactance* limits the sustained short-circuit current. By the time sustained conditions are reached, the field current, if controlled by an automatic regulator, would probably be considerably increased. Decrement curves, plotted from elaborate series of short-circuit tests, are much used. (See reference at end of chapter.)

Negative-phase-sequence and zero-phase-sequence armature currents do not set up magnetomotive forces which rotate at the same speed and direction as the main field, and so there is for these components no distinction between direct-axis and quadrature-axis values.

Typical values of the different reactances and time constants for various types of synchronous machines have been determined by R. H. Park and B. L. Robertson (see reference at end of chapter), and their data are reproduced in table XXIV.

Transient direct-current components of armature currents on short circuit may have initial magnitudes as large as the peak of the alternating current.

The symbols in table XXIV are defined as follows:

x_d = direct-axis synchronous reactance.
x_q = quadrature-axis synchronous reactance.
x_d' = direct-axis transient reactance.
x_q' = quadrature-axis transient reactance.
x_d'' = direct-axis subtransient reactance.
x_q'' = quadrature-axis subtransient reactance.
x_2 = negative-phase-sequence reactance.
x_0 = zero-phase-sequence reactance.
T_0 = open-circuit time constant.

The time constant T for a short circuit through an external reactance x is

$$T = \frac{x_d' + x}{x_d + x} T_0 \tag{447}$$

The zero-sequence reactance is in effect infinite if there is no neutral connection to the machine capable of carrying current.

TABLE XXIV*

Typical Values of Per-Unit Reactances and Time Constants for Synchronous Machines

	x_d	x_q	x_d'	x_q'	x_d''	x_q''	x_2	x_0†	T_0 (sec)
Synchronous motors									
High-speed......	0.65–0.80–0.90	0.50–0.65–0.70	0.15–0.25–0.35	0.50–0.65–0.70	0.10–0.18–0.25	0.12–0.20–0.30	0.11–0.19–0.25	0.02–0.15	2–4
Low-speed........	0.80–1.1–1.5	0.60–0.80–1.10	0.40–0.50–0.70	0.60–0.80–1.1	0.25–0.35–0.45	0.30–0.40–0.50	0.25–0.35–0.50	0.04–0.27	2–4
Synchronous Condensers...........	av. 1.60	av. 1.00	0.40–0.50	av. 1.00	0.25–0.30	0.25–0.35	0.25–0.32	0.04–0.10	5–7
Water-wheel Generators	0.60–1.0–1.25	0.40–0.65–0.80	0.20–0.35–0.45	0.40–0.65–0.80	0.15–0.22–0.35	0.40–0.65–0.80	0.25–0.45–0.60	0.02–0.21	3–6
Steam turbogenerators									
Solid-rotor........	av. 1.15	av. 1.0	0.15–0.25	0.15–0.25	0.08–0.15	0.10–0.20	0.08–0.13	0.01–0.08	4–7
Laminated-rotor...	av. 1.15	av. 1.0	0.15–0.25	av. 1.0	0.08–0.12	0.20–0.30	0.11–0.21	0.01–0.08	4–7

* Park and Robertson.

† x_0 varies from about 15 per cent to 60 per cent of x_d'', depending upon winding pitch.

112. Short-Circuit Currents in Systems. Let figure 136 represent one link in a balanced three-phase system, upon which a short circuit is to take place. It will be convenient to bring out three hypothetical leads *a*, *b*, *c*, of assumed zero impedance, upon which to impress the fault.

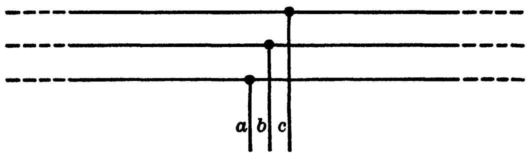

FIG. 136. One link in a balanced three-phase system.

Single-line-to-ground fault. Assume that a single-line-to-ground fault takes place on phase *a*. The current which will flow will be due to all the generated voltages in the system, and computation of the component produced by each, and final summation, would be very tedious.

Consider the connection shown in figure 137. A hypothetical generated voltage E_a, equal and opposite to the voltage to ground of phase *a* before short circuit, is shown. If a fault is produced by closing switch *S*, there will be no current flow because the voltages are in

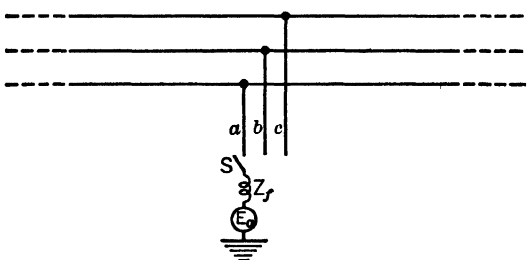

FIG. 137. Hypothetical generator in series with single-line-to-ground fault.

balance. Therefore, by the principle of superposition, the component of current at the fault and at every other point in the system, due to the hypothetical generator, will be just equal and opposite to the sum of all the fault currents produced by the actual generated voltages throughout the system.

The fault current is then the same as the current that would be pumped into a system having no generated voltages, but unchanged impedances, by our hypothetical generator of voltage E_a.

Let the system impedances to positive-, negative- and zero-sequence currents, looking into the system from the point of fault, be Z_1, Z_2 and Z_0 respectively. The fault currents in phases *b* and *c* are zero, and if we resolve the fault currents in a, b and c into their symmetrical components, as illustrated in figure 138, we find that

$$I_{a1} = I_{a2} = I_{a0} = \frac{I_a}{3}. \tag{448}$$

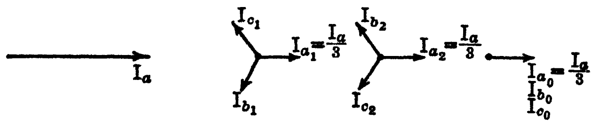

FIG. 138. Resolution of currents due to single-line-to-ground fault into their symmetrical components.

We may write, letting Z_f represent the impedance at the point of fault,

$$I_aZ_f + I_{a1}Z_1 + I_{a2}Z_2 + I_{a0}Z_0 = E_a, \tag{449}$$

and so

$$I_a = \frac{3\,E_a}{Z_1 + Z_2 + Z_0 + 3\,Z_f}. \tag{450}$$

$$I_{a1} = \frac{E_a}{Z_1 + Z_2 + Z_0 + 3\,Z_f}. \tag{450a}$$

In a dead short circuit, Z_f is equal to zero. Equation (450) is the same as the expression for current in a circuit having generated voltage $3\,E_a$ in series with four impedances as given in the denominator.

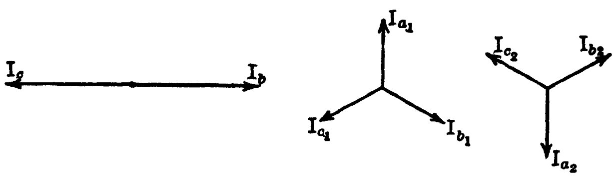

FIG. 139. Resolution of currents due to line-to-line fault into their symmetrical components.

Line-to-line fault. If the fault occurs between two phases b and c, then the phase currents at the point of fault are given by:

$$I_b = -I_c.$$
$$I_a = 0.$$

The symmetrical components of such a set of currents are shown in figure 139. There is no zero-sequence component, and $I_{a1} = -I_{a2} = \frac{I_b}{\sqrt{3}}\underline{|90°}$. The original voltage E_{bc} before the fault was equal to $\sqrt{3}\,E_a\underline{|90°}$. Proceeding as before, we can equate

$$\begin{aligned} -E_{bc} = -E_b + E_c &= I_{b1}Z_1 + I_{b2}Z_2 - I_{c1}Z_1 - I_{c2}Z_2 + I_bZ_f \\ &= I_b(Z_1 + Z_2 + Z_f). \end{aligned} \tag{451}$$

Therefore

$$I_b = -\frac{E_{bc}}{Z_1 + Z_2 + Z_f}; \tag{452}$$

or

$$I_{a1} = \frac{E_a}{Z_1 + Z_2 + Z_f}. \tag{453}$$

Double-line-to-ground fault. In the double-line-to-ground fault it is not so immediately apparent what will be the relations among the phase values of short-circuit current but the equations governing them are readily written. Let it be assumed that there is no impedance in the fault between the two short-circuited lines b and c, but that there is an impedance Z_f between the two lines and ground. Equating to zero the voltage around the loop through phase b of the system and neutral; and similarly for the loop through c and neutral,

$$E_b - I_{b1}Z_1 - I_{b2}Z_2 - I_0Z_0 = 0, \tag{454}$$

$$E_c - I_{c1}Z_1 - I_{c2}Z_2 - I_0Z_0 = 0. \tag{455}$$

Solving,

$$I_{a1} = \frac{E_a}{Z_1 + \dfrac{Z_2(Z_0 + 3\,Z_f)}{Z_2 + Z_0 + 3\,Z_f}}, \tag{456}$$

$$I_{a0} = -\frac{I_{a1}Z_2}{Z_2 + Z_0 + 3\,Z_f}, \tag{457}$$

$$I_{a2} = -I_{a1} - I_{a0}. \tag{458}$$

Example. The power system shown in figure 140 has a dead short circuit at the midpoint A of the line. Find the short-circuit current for (1) a single-line-to-ground fault, (2) a line-to-line fault and (3) a double-line-to-ground fault.

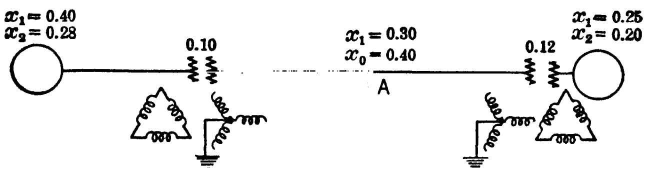

FIG. 140. Power system with two generating stations, two transformer banks, and one transmission line. Reactances are given in per-unit quantities.

Solution. The positive-phase-sequence impedance looking into the system from point A is

$$Z_1 = \frac{(0.40 + 0.10 + 0.15)\ (0.15 + 0.12 + 0.25)}{0.65 + 0.52}$$

$$= 0.289 \text{ per unit.}$$

$$Z_2 = \frac{(0.28 + 0.10 + 0.15)\ (0.15 + 0.12 + 0.20)}{0.53 + 0.47}$$

$$= 0.249 \text{ per unit.}$$

$$Z_0 = \frac{(0.10 + 0.20)\ (0.20 + 0.12)}{0.30 + 0.32}$$

$$= 0.155 \text{ per unit.}$$

For the single-line-to-ground fault, by (450), the current through the fault, on a per-unit basis, is

$$I_f = \frac{3}{0.289 + 0.249 + 0.155} = 4.32.$$

For the line-to-line fault, by (452), the size of current in the fault is

$$I_f = \frac{\sqrt{3}}{0.289 + 0.249} = 3.22.$$

For the double-line-to-ground fault, by (456), (457) and (458)

$$I_{a1} = \frac{1}{0.289 + \dfrac{0.249 \times 0.155}{0.249 + 0.155}} = 2.60,$$

$$I_{a0} = -\frac{2.60 \times 0.249}{0.249 + 0.155} = -1.60,$$

$$I_{a2} = -2.60 + 1.60 = -1.00.$$

The fault current in phase b is

$$I_b = 2.60\ \underline{|-120^\circ} - 1.60 + 1.00\underline{|-60^\circ}$$

$$= -2.40 - j\,3.12 = 3.93\underline{|-127.6^\circ}.$$

$$I_c = 2.60\underline{|120^\circ} - 1.60 + 1.00\ \underline{|60^\circ}$$

$$= -2.40 + j\,3.12 = 3.93\ \underline{|127.6^\circ}.$$

$$I_g = 3\,I_0 = -4.80\underline{|0^\circ}.$$

Short-circuit currents at other points in the system. To determine values of short-circuit current at points in the system remote from the fault, it is convenient to use the equivalent circuits illustrated in figures 141, 142 and 143. A comparison of these circuits with the corresponding equations will show their equivalence. Note that in figure 142 the

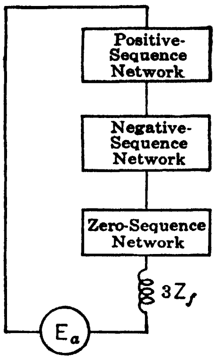

FIG. 141. Equivalent circuit for computing symmetrical components of current in phase a with single-line-to-ground fault on phase a.

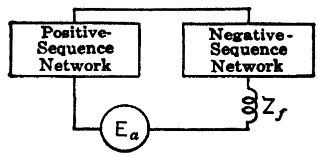

FIG. 142. Equivalent circuit for computing symmetrical components of current in phase a with line-to-line fault between phases b and c.

current flowing upward through the positive-sequence network must flow downward through the negative-sequence network. Thus the difference in sign of I_{a1} and I_{a2}, as indicated in figure 139, is taken into account in the equivalent circuit. The short-circuit current which

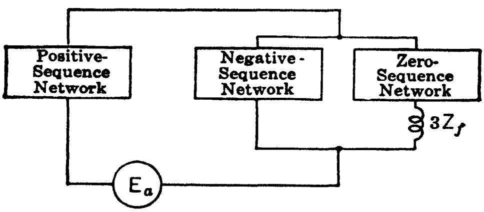

FIG. 143. Equivalent circuit for computing symmetrical components of current in phase a with double-line-to-ground fault in phases b and c.

will flow in any circuit element in phase a is the sum of the positive-, negative- and zero-sequence components flowing in the respective networks in that circuit element. For phase b, the component currents are those of phase a rotated -120 degrees, $+120$ degrees and 0 degrees respectively, for the positive, negative and zero sequences. For phase *c*, the corresponding rotations are +120 degrees, −120 degrees and 0 degrees.

The equivalent circuit for the system of figure 140, for a single-line-to-ground fault at point *A*, is shown in figure 144.

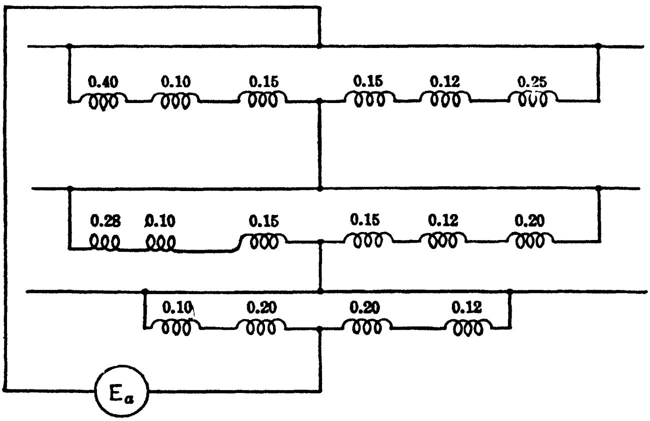

FIG. 144. Equivalent circuit for system of figure 140, for single-line-to-ground fault.

113. Use of Calculating Boards or Model Systems. Calculating boards or model systems are used extensively in short-circuit calculations. By the use of equivalent circuits for the different symmetrical-component networks, as illustrated in figures 141–144, it is possible to represent unsymmetrical three-phase short circuits on a single-phase board. Direct-current calculating boards give very good results for short-circuit work. The resistances of the main system circuit elements are usually so small compared to the inductive reactances that phase differences among the currents in the various elements due to this cause are very small on short circuit. In the model, the reactances in the system are represented to scale by resistances.

With alternating-current calculating boards, elements of resistance, inductance and capacitance are available, so that an exact replica of the impedances of any system may be made, provided that the capacity of the board is not exceeded in respect to the number of elements.

114. Simplification of Networks. A common assumption in making an approximate calculation of the current flowing into a fault from a system which contains a number of generating stations is that all the excitation voltages of the generators and synchronous condensers are equal and in phase. They are then considered as being all in parallel and feeding the fault through the system network. The system may be simplified by Y-Δ and other such transformations, to the point where only a single impedance or reactance (if resistances are being neglected) remains between the composite generating station and the fault.

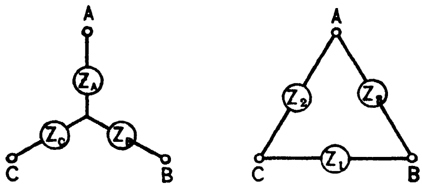

FIG. 145. Y and Δ connections.

For calculating initial fault current the synchronous condensers are included as generators, but for sustained fault current they are omitted.

The formulas for Y-Δ and Δ-Y transformations, with the meanings of the symbols as indicated in figure 145, may be stated as follows:

The external electrical characteristics of the Y and the Δ of figure 145 are identical for the fundamental frequency and in the steady state if

$$Z_A = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3}, \tag{459}$$

$$Z_B = \frac{Z_1 Z_3}{\text{same denominator}}, \tag{460}$$

$$Z_C = \frac{Z_1 Z_2}{\text{same denominator}} \tag{461}$$

The Δ impedances in terms of the Y impedances are

$$Z_1 = \frac{Z_A Z_B + Z_A Z_C + Z_B Z_C}{Z_A}, \tag{462}$$

$$Z_2 = \frac{\text{same numerator}}{Z_B}, \tag{463}$$

$$Z_3 = \frac{\text{same numerator}}{Z_C}. \tag{464}$$

The Y-Δ transformation is a particular case of a general star-mesh transformation which may be stated as follows:

The external characteristics of the n-point star and n-point mesh are identical between the n terminal points $A, B, C, D, \cdots N$ if

$$AB = ab\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \cdots + \frac{1}{n}\right) \tag{465}$$

$$BC = bc \times \text{same summation,} \tag{466}$$
$$AC = ac \times \text{same summation,} \tag{467}$$

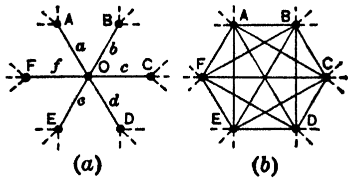

Fig. 146. General star and mesh networks.

and so on, in which the symbols (refer to figure 146) have the following meanings:

a = impedance from A to O in vector ohms, in the star connection,
b = impedance from B to O in vector ohms, in the star connection,
c = impedance from C to O in vector ohms, in the star connection,
AB = impedance from A to B in vector ohms, in the mesh connection,
BC = impedance from B to C in vector ohms, in the mesh connection,
AC = impedance from A to C in vector ohms, in the mesh connection,

and so on.

These formulas are sometimes useful in solving network problems.

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PROBLEMS ON CHAPTER XI

Prob. 1–11. Calculate the short-circuit current which will flow in a single-line-to-ground fault occurring in the system of figure 140, at the right-hand end of the transmission line.

Prob. 2–11. Calculate the short-circuit current which will flow in a line-to-line fault occurring in the system of figure 140, across the terminals of the left-hand generator.

Prob. 3–11. The system of figure 140 is altered by connecting a transmission line of $X_1 = 0.20$ and $X_0 = 0.30$ to point A; and the other end of this line, through a grounded Y-Δ transformer bank of $X = 0.10$, is connected to a synchronous machine of $X_1 = 0.25$ and $X_2 = 0.18$. Solve for the single-line-to-ground fault current at A.

Prob. 4–11. In the circuit of problem 3–11, find the line-to-line fault current, for fault at A.

Prob. 5–11. In the circuit of problem 3–11, find the double-line-to-ground fault currents, for fault at A.

Prob. 6–11. In the circuit of problem 3–11, there is a line-to-line fault at the left-hand generator. Find the three-phase currents in each of the other two generator leads

Prob. 7–11. Find all three generator currents for the conditions of problem 3–11.

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